Category Theory 1

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1 Categories: definitions and examples

A collection of objects is just a collection of objects. You can't exactly *do* anything with those objects, mathematically speaking, unless you know a bit more about what the objects *are* – or at least, how they're related to each other. The idea of a *category* is to give a crucial bit more information so that we can begin to study the structures involved, rather than just staring at them as *things*.

Often the starting point in Category Theory is to look around us in the world – the mathematical world, that is – and spot the patterns. A lot of mathematics is a sort of fancy kind of pattern-spotting, actually. In Category Theory we go round looking for common scenarios that seem to arise all over the place in slightly different guises. We turn that "common scenario" into a piece of Category Theory. The definition of *category* is a case in point.

Let's start by thinking about some mathematical notions.

1.1 Some examples to get us going

Groups

A *group* is a set X equipped with a binary operation $\circ$ and an identity element e, satisfying some axioms:

- associativity: for all $a, b, c \in X$, $(a \circ b) \circ c = a \circ (b \circ c)$,

- unit laws: for all $a \in X$, $e \circ a = a = a \circ e$,
- inverses: for all $a \in X$ there is an element a^{-1} such that

$$a \circ a^{-1} = e = a^{-1} \circ a.$$

We can classify this information as:

- Data – a set X
- Structure – a binary operation and unit
- Properties – associativity and unit axioms, and existence of inverses.

Now, we also have a notion of "map of groups" $X \longrightarrow Y$, that is, *group homomorphism*. Fundamentally it's a function on the underlying sets of X and Y, but it also has to interact sensibly with the structure of the groups X and Y, otherwise it would be a bit silly. What does "interact properly" mean? Well, what's the "structure" we're supposed to be thinking about here? We consult our Data-Structure-Properties list, and we see that our structure is: binary operation and unit[1].

Now, if we have elements a and b in X and a function F, there are two different things we could do:

1. We can do the binary operation $\rightsquigarrow$ $a \circ b$
   and then apply the function F $\rightsquigarrow$ $F(a \circ b)$.
2. We can apply F to each element $\rightsquigarrow$ Fa, Fb
   and then do the binary operation $\rightsquigarrow$ $Fa \circ Fb$.

When we say "F interacts properly with $\circ$" we mean that these two things produce the same result, i.e.,

$$F(a \circ b) = Fa \circ Fb.$$

Similarly we need F of the unit element in X to be the unit element in Y.

We can think of all this as:

- F is sensible
- F interacts properly with the group structure of X
- F respects the group structure of X
- F preserves the group structure of X
- F is structure-preserving.

Do we have anything similar going on with other mathematical notions?

[1] You might think that inverses are "structure" as well, but you can check that if a function respects the binary operation and identity then it automatically respects the inverses as well

Rings

A ring is a group with even more *stuff.* It's a group under addition (the binary operation is usually written as addition, anyway), and a monoid under multiplication – so we can multiply but we can't divide. So we have

- Data – a set X
- Structure – $+, 0, \times 1$
- Properties – group axioms, monoid axioms and a distributive law.

So a *ring homomorphism* is a function that *preserves* the structure listed:

- $F(a + b) = Fa + Fb$
- $F(a \times b) = Fa \times Fb$
- $F(0) = 0$
- $F(1) = 1$.

Monoids

Now that we've mentioned monoids, we might as well think about them for a second. A monoid is given by:

- Data – a set X
- Structure – a binary operation $\circ$ and unit 1
- Properties – associativity and unit axioms.

So a monoid is like a group but without necessarily having inverses. For example the natural numbers $\mathbb{N}$ (including 0) form a monoid under addition, but not a group[2].

Now a monoid homomorphism is just like a group homomorphism – we must have

$$F(a \circ b) = Fa \circ Fb$$

and

$$F(1) = 1.$$

As usual, it is a structure-preserving map.

[2] They also form a monoid under multiplication, but later we'll see a precise way in which this is less canonical a structure.

Vector spaces

A vector space is also a group with some extra structure – scalar multiplication[3]. We start with a ground field k of scalars, and then a vector space over k has:

- Data – a set X
- Structure – a group operation $+$, unit 0 and scalar multiplication
- Properties – group axioms, distributive laws, associativity of scalar multiplication.

Then a structure-preserving map has to preserve addition, unit, and scalar multiplication. And that's exactly the definition of a linear map.

Posets

A poset is given by

- Data – a set X
- Structure – a relation $\leq$
- Properties – transitivity, reflexivity

Then a structure-preserving map is one that preserves the relation $\leq$, that is

$$x \leq y \Rightarrow Fx \leq Fy.$$

So what we have here is an order-preserving map.

Sets

We haven't mentioned the most obvious example so far – sets. Sets are, erm, sets with no extra structure. So the "structure-preserving maps" are just functions – there's nothing here to preserve.

Question: What can we do with these structure-preserving maps?

What we've done so far is *recognise* some structure-preserving maps; but what can we *do* with them? We can always compose them – if we have two such maps meeting in the middle

$$X \xrightarrow{F} Y \xrightarrow{G} Z$$

we can produce a composite

$$G \circ F : X \longrightarrow Z.$$

[3] You may think that a vector space is just $\mathbb{R}^n$ for some n. You have a point – every finite-dimensional real vector space is *isomorphic* to an $\mathbb{R}^n$, and we'll use this fact later when we're re-expressing the category of vector spaces linear maps using natural numbers and matrices.

We also have an identity map

$$X \xrightarrow{1_X} X$$

which leaves all the data exactly the same – after all, it would be rather stupid if "doing nothing" did not count as "preserving the structure".

Moreover, the composition is *associative* – if we have three maps

$$X \xrightarrow{F} Y \xrightarrow{G} Z \xrightarrow{H} U$$

we know that

$$(H \circ G) \circ F = H \circ (G \circ F).$$

This information is exactly what we encode in the notion of a *category*.

1.2 The definition of category

There are many equivalent ways of defining a category, with varying levels of abstraction and intuitive-ness. We will start with just one definition, and will mention other characterisations later. A certain amount of formality is useful for developing the theory; the hope is that the intuition is captured in the formalism.

Summarised informal definition

A *category* $\mathcal{C}$ consists of

- Data – objects and morphisms
- Structure – composition and identities
- Properties – associativity and unit axioms

Definition 1.1. A *category* $\mathcal{C}$ consists of:

- a collection $\text{ob}\mathcal{C}$ of objects, and
- for every pair $X, Y \in \text{ob}\mathcal{C}$, a collection $\mathcal{C}(X, Y)$ of morphisms $f : X \longrightarrow Y$,

equipped with

- for each $X \in \text{ob}\mathcal{C}$, an identity morphism $1_X \in \mathcal{C}(X, X)$, and
- for all $X, Y, Z \in \text{ob}\mathcal{C}$, a composition map

$$\begin{array}{rccc} m_{XYZ} & : & \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) & \longrightarrow & \mathcal{C}(X,Z) \\ & & (g, f) & \mapsto & g \circ f \end{array}$$

satisfying the following axioms:

- unit laws – given $f : X \longrightarrow Y$ we have $1_Y \circ f = f = f \circ 1_X$, and
- associtivity – given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.

A category is said to be *small* if $\text{ob}\mathcal{C}$ and all of the $\mathcal{C}(X, Y)$ are not just collections but actually sets, and *locally small* if each $\mathcal{C}(X, Y)$ is a set.

Notes

1. We'll discuss this "sets vs collections" issue in the Aside below.

2. If $f \in \mathcal{C}(X, Y)$ we say that X and Y are the *domain* and *codomain* of f, or the *source* and *target*. Sometimes we prefer the latter because the former sounds too much like we're talking about a function or map, and morphisms might be nothing at all like functions or maps.

3. Morphisms are also referred to as *maps* (although they might be nothing like maps) or *arrows*.

4. The collection of arrows $\mathcal{C}(X, Y)$ is sometimes also written $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, and if it's a set we refer to it as a *hom-set*. We can also write $\mathrm{Hom}_{\mathcal{C}}$ for the collection of all morphisms of $\mathcal{C}$, or alternatively $\mathrm{Mor}\mathcal{C}$ or $\mathrm{Arr}\mathcal{C}$.

5. As usual, we often write gf for $g \circ f$ when we get bored of writing the little circle.

1.3 Aside on set theory

The main thing we need to remember about set theory is that some collections of things count as sets and some don't. If you haven't done much set theory you might wonder what this can possibly mean – isn't a set literally just a collection of things? How can some collections count as sets and not others? The answer is – informally yes, a set is just a collection of things, and this is what "set" means in primary school, in normal life, and also when you first do sets and functions as undergraduates. Unfortunately it's one of those things you later have to unlearn – in advanced mathematics, the definition of a set is much more subtle. Why? The answer has to do with trying to think about *the set of all sets*, and Russell's paradox.

> *The barber shaves everyone in the town who doesn't shave himself, and nobody else. Who shaves the barber?*

This question should make you feel dizzy. If the barber shaves himself, that means he doesn't shave himself. . .and if he doens't shave himself, that means he does. . .

This is the essence of Russell's paradox, although in fact Russell's paradox is about *sets which are members of themselves*. How on earth can a set be a member of itself? Well, consider this set:

$$X = \text{the set of all sets.}$$

In this case, X is a set so X must be a member of X. Now consider this set:

$$S = \text{the set of all sets that are members of themselves.}$$

Now certainly $X \in S$. Is S a member of S? It's rather hard to tell – it could go either way. Here's a worse set:

$R =$ the set of all sets that are *not* members of themselves.

Certainly X is not a member of R. But is R a member of R? If it isn't a member of itslef, then by definition of R it is supposed to be in R. If it isn't a member of itself then by definition of R it *is* supposed to be in R. Help!

This is Russell's paradox, and warns us that we need to be rather careful about the things that are allowed to be sets. Basically we have to avoid things like "the set of all sets". Intuitively this is too "large" to be a set and is referred to as a collection or "proper class". Sets are special collections that can be built up only using certain carefully controlled principles, crucially avoiding any self-reference or mentions of self-membership that get us into trouble with the paradox. Making this precise was the major achievement of axiomatising set theory. It's like saying a suitcase is only properly packed if things have been put in carefully, not thrown in willy-nilly.

For our purposes we can think:

Sets are small, classes are large.

The definition of small category prevents a Russell-like paradox arising from considering the "category of all categories". We'll later see that the category of small categories is large, so Russell's paradox is avoided - it's isn't a small category, so can't be a member of itself. Later, notions of small and large will be important when we study limits and colimits – some categories will have small limjits but not large ones.

Here endeth the aside.

1.4 Examples of categories

Large categories of mathematical structures

These are the ones we've already seen; here are some more as well. You may or may not have met these mathematical structures before, but it doesn't particularly matter for now. We give these categories a name which usually refers to the objects of the category without much reference to the morphisms, except in some cases where the morphisms are somehow the things we're studying and the objects are just there supporting them, such as **Rel** and **Mat**.

1. **Set** of sets and functions. This is in many ways the "prototype category". We often look at familiar constructions involving sets and see if we can express them entirely "categorically", so that we can then look for similar features in other categories. Expressing something "categorically" here means expressing the construction using only the language of objects and morphisms, which in particular for sets means *never referring to the elements of a set*. **Set** has many, many wonderful features that make it a good place to start doing mathematics, and we would like to know what other categories have these features.

2. Categories derived from or related to **Set**:

   - **Pfn** of sets and partial functions; these are functions that are not defined everywhere on their domain set.
   - **Rel** of sets and relations; a morphism $A \longrightarrow B$ is thus a subset of $A \times B$ giving the pairs (a, b) such that $a \sim b$. Given a relation on A and B, and a relation on B and C we produce the "composite" relation on A and C which has $a \sim c \iff \exists b \in B$ such that $a \sim b$ and $b \sim c$. Later ***ref we'll look at some very interesting features of **Rel**, and we'll see how we can take equivalence classes in an entirely categorical way. For now, it's worth observing that given any relation on A and B we get a morphism $A \longrightarrow B$ but also a morphism $B \longrightarrow A$, and these morphisms are somehow expressing the same thing. We will later have a way of expressing precisely what is going on here ***ref.
   - $\mathbf{Set}_*$ of pointed sets and basepoint-preserving functions. A pointed set is just a set with a chosen element as its "basepoint"; the morphisms are functions which send basepoints to basepoints. Note that the empty set is not an object in this category – the "smallest" possible pointed set has one element (which must therefore be the basepoint).

3. Algebraic structures and structure-preserving maps: these are all "sets with some extra structure" and we'll study this sort of situation abstractly when we do monads and their algebras. The notion of "set with extra structure" turns out to be so ubiquitous that it is worth studying it as a concept in its own right, and indeed generalising it somewhat. Various standard constructions then turn out to have categorical explanations, for example the expression of a group by generators and relations.

   - **Grp** of groups and group homomorphisms;
   - **Ab** of abelian groups and group homomorphisms;
   - **Ring** of rings and ring homomorphisms;
   - **Vect** of vector spaces over $\mathbb{R}$; to be more precise we might want to call this $\mathbf{Vect}_{\mathbb{R}}$ because of course we might want to consider the category $\mathbf{Vect}_k$ of vector spaces over some other field, k. This category is another very rich category with many fruitful structures that we wish to isolate and look for in other categories. One interesting feature is that the difference between finite-dimensional vector spaces and infinite-dimensional ones becomes a key categorical concept.
   - **Field** of fields and field homomorphisms. This actually turns out not to be a very well-behaved category, and it is often better to restrict to fields of a fixed characteristic.

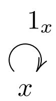
4. Categories of topological spaces: these were among the earliest motivating examples of categories. Using this framework gives very concise statements of results in algebraic topology. For example the fundamental group construction becomes a functor (as do homology and cohomology), Van Kampen's theorem is a matter of preservation of colimits, the covering space correspondence becomes an equivalence of categories – these are all notions that we will meet later on.

    - **Top** of topological spaces and continuous maps; it is worth noting that unlike **Set** and **Vect**, **Top** is not a well-behaved category. For this reason we often end up working in some related category of "better" spaces, to eliminate various pathologies that can otherwise arise and ruin everything.
    - **Haus** of Hausdorff spaces and continuous maps;
    - **CHaus** of compact Hausdorff spaces and ***; this is a much better category than **Top**.
    - **Met** of metric spaces and uniformly continuous maps;
    - **Htpy** of topological spaces and homotopy classes of maps; this category is sometimes preferable to **Top** (even to better versions of **Top**) because we are often interested in things "up to homotopy", and this isn't something that the morphisms in **Top** can express. For example, we are often much more interested in the notion of "homotopic" spaces than "homeomorphic" spaces, because the latter is too restrictive to be useful. Doing things "up to homotopy" is a big business, and the relationship between **Top** and **Htpy** is a very interesting and worthy prototype.

1.4.1 "Very" small categories

When categories are very small indeed[4] we can quite simply and vividly draw them on the page using points for objects and arrows for morphisms.

1. There is a category with one object and one morphism. Since there is only one morphism, we know it simply has to be the identity. We can draw a picture of this category:



where of course it doesn't matter whether we called the single object x or anything else. (In practise we quite often call it $*$, to capture the fact that it's essentially a nameless object which either has no distinguishing features, or else we're ignoring them because we just don't care.) This

[4] This is not technical terminology.

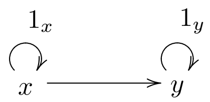
category is often referred to as **1**. You might think it's the tiniest and most stupid possible category – but the next one is even "tinier". In fact both this and the next category have universal properties, which we'll see later ***ref, showing that they're each superlative in some specific way.

2. There is an empty category. It has no objects and no morphisms, so must surely be the "tiniest" and most stupid category. Still, it's not completely worthless and is often called **0**, and after all, the number 0 is very far from useless. We could try to draw the category **0** on the page, but nothing much would happen.

3. There is a category that we might draw like this:

$$\cdot \longrightarrow \cdot$$

which means there are two objects and one morphism between them, as shown. (We don't nessarily label the objects, if we're feeling lazy and it doesn't seem important what those two objects are called.) Of course, this picture isn't *entirely* honest – there must be identities as well, so realy this category has three morphisms and could be drawn like this:



but we don't usually bother drawing in the identities because we know that they have to be there whether we draw them or not.

4. Here's another very small category, albeit it slightly larger than the last one:

$$\cdot \rightrightarrows \cdot$$

which really has 4 morphisms if we count the undrawn identities. The two morphisms drawn are referred to as "parallel" – they share the same source and target and so *look* parallel, at least in this picture.

Note that in all the above examples there was no non-trivial composition – the only composition involved the identity so didn't tell us anything interesting. The next examples will be a smigden more interesting.

5. Here's a category with some non-trivial composition:

$$\cdot \longrightarrow \cdot \longrightarrow \cdot$$

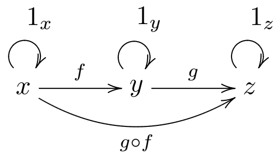
but we now have to be a bit more careful about how we read this picture. For the sake of explaining it, let's give it some labels:

$$x \xrightarrow{f} y \xrightarrow{g} z$$

and first of all note that we have as usual omitted to draw in the identity arrows 1_x, 1_y, 1_z. But we also know that there must be a composite

$$x \xrightarrow{g \circ f} z$$

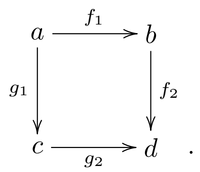
which we have also omitted, on the grounds that it simply has to be there. So really this category looks like this:



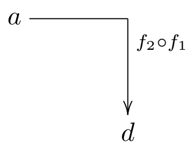
but hopefully you'll agree that this picture is rather more cluttered and doesn't actually tell us any more than the first picture.

What we've actually done here is use the first diagram to *generate* a category freely. This is a bit like expressing a group in terms of generators and relations, except that we didn't have any relations – but that was only because the category wasn't very complicated. In the next example we'll also have some relations.

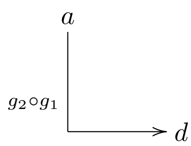
6. There is a category defined by the *commutative square*:



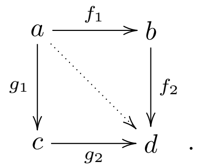
First as usual we have an identity on each object. We also have composites



and



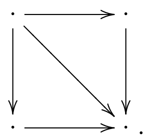
and each of these composites could in fact be drawn as a diagonal through the original square:



Now, saying the square commutes means that "going round the square in different ways gives the same answer", that is, that the two composites are equal:

$$f_2 \circ f_1 = g_2 \circ g_2.$$

This is a *relation* that we impose on our generators for this category; later we'll have a way of making precise this "generators-and-relations" description of a category[5] and, in particular, giving a precise sense in which it's the same idea as the "generators-and-relations" description of a group. In summary, the above category has the following non-identity arrows:



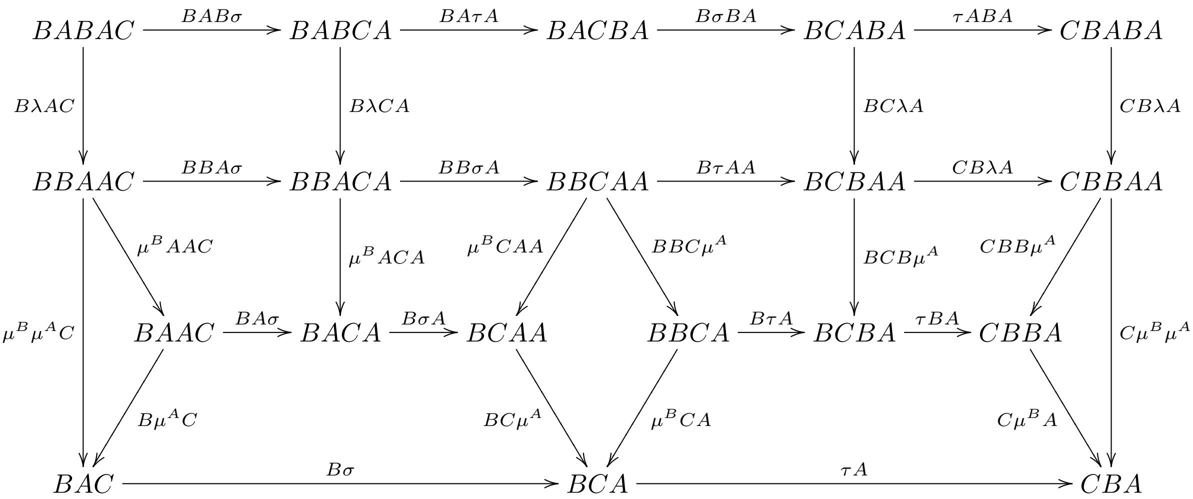
Aside on commutative diagrams

This diagrammatic representation of equalities between morphisms is terribly useful, and highly characteristic of category theory[6]. Often the aim is to show that a rather larger diagram commutes, and we proceed by "filling in" the diagram with some smaller diagrams that we know commute. For example[7]:

[5] We are starting with generators given by a directed graph; we form the free category on this graph; we impose equalities between some paths with the same domain and codomain.

[6] For some people this is their overriding impression of category theory – commutative diagrams.

[7] This is example is taken from ***cite



This could of course be written out as a series of substitutions in equations, but often the geometric nature of the diagrams helps us to see what substitutions are going to provide the result. It also gives us a vivid way of keeping track of which substitution we have performed at each stage, arguably rather more vivid than a string of symbols written in a line. The above diagram written out in equations looks like this[8]:

$$
\begin{aligned}
&C\mu^B\mu^A \circ CB\lambda A \circ \tau ABA \circ B\sigma BA \circ BA\tau A \circ BAB\sigma \\
&\quad = C\mu^B A \circ CBB\mu^A \circ CB\lambda A \circ \tau ABA \circ B\sigma BA \circ BA\tau A \circ BAB\sigma \\
&\quad = C\mu^B A \circ CBB\mu^A \circ \tau BAA \circ BC\lambda A \circ B\sigma BA \circ BA\tau A \circ BAB\sigma \\
&\quad = C\mu^B A \circ CBB\mu^A \circ \tau BAA \circ B\tau AA \circ BB\sigma A \circ B\lambda CA \circ BAB\sigma \\
&\quad = C\mu^B A \circ CBB\mu^A \circ \tau BAA \circ B\tau AA \circ BB\sigma A \circ BBA\sigma \circ B\lambda AC \\
&\quad = C\mu^B A \circ \tau BA \circ BCB\mu^A \circ B\tau AA \circ BB\sigma A \circ BBA\sigma \circ B\lambda AC \\
&\quad = C\mu^B A \circ \tau BA \circ B\tau A \circ BBC\mu^A \circ BB\sigma A \circ BBA\sigma \circ B\lambda AC \\
&\quad = \tau A \circ BC\mu^A \circ \mu^B CAA \circ BB\sigma A \circ BBA\sigma \circ B\lambda AC \\
&\quad = \tau A \circ BC\mu^A \circ B\sigma A \circ \mu^B ACA \circ BBA\sigma \circ B\lambda AC \\
&\quad = \tau A \circ BC\mu^A \circ B\sigma A \circ BA\sigma \circ \mu^B AAC \circ B\lambda AC \\
&\quad = \tau A \circ B\sigma \circ B\mu^A C \circ \mu^B AAC \circ B\lambda AC \\
&\quad = \tau A \circ B\sigma \circ \mu^B\mu^A C \circ B\lambda AC
\end{aligned}
$$

Curious categories

The following examples are a bit curious and might surprise you a bit. But they're a central part of category theory and yield all sorts of interesting constructions.

1. Every set "is" a category with only identity morphisms. We have

[8]In case you're wondering, typesetting this calculation was a bit painful, and no, I'm not really sure if I got it right.

$$\mathcal{C}(X, Y) = \begin{cases} \{1_X\} & \text{if } X = Y \\ \emptyset & \text{otherwise} \end{cases}$$

Such a category is called *discrete*. You might think it's a bit stupid to express a set in this way but it's useful to have sets fitting into our framework too. The next examples involve the "opposite" situation – instead of trivial morphisms, we'll have trivial objects.

2. Every group "is" a category with only one object. That is to say we can express a group G as a category:

   - it has only one object, which we might call $*$;
   - the morphisms $* \longrightarrow *$ are precisely all the elements of G;
   - composition is group multiplication, and the identity 1_x in the category is the unit element of the group G.

   This example might make you feel a bit queasy at first, and that's perfectly normal, so don't worry about it. Try imagining that the group is a symmetry group of an object, say, a cube. Then you can imagine a category with one object – the cube itself. The morphisms are the various reflections, rotations, and combinations of them, each of which sends the cube to itself, albeit a different way up. When you first met groups they were probably all symmetry groups; later you were asked to imagine more abstract groups that weren't obvious geometric symmetries of some object. Likewise we go more abstract with our category – the single object isn't really the object whose symmetries we are taking – it's just a formal object, to get the category going[9].

   Note that every group gives rise to a category with only one object but not every category with only one object is a group; that is our next example.

3. A category with only one object "is precisely[10]" a monoid. Of course, every group is in particular a monoid, and so every group is a category with one object as in example 1. However, being a group involves the added condition that every element has an inverse. We'll define inverses for morphisms in a category later[11] A monoid is like a group but doesn't necessarily have inverses for all its elements.

   Given a monoid M, we can express it as a category:

   - there is only one object $*$, as for groups;
   - the morphisms $* \longrightarrow *$ are precisely the elements of M;
   - composition is multiplication in the monoid, and the identity morphism is the unit of the monoid.

[9] This framework gives a concise and fruitful characterisation of group representations, as we'll see later ***ref

[10] We can make this precise once we know what "equivalence of categories" is.

[11] Perhaps you can already work out what they are.

So far this is just the same as for groups. The single object is completely "formal" and just serves to tell us that any morphism can be composed with any other, so composition is really just a binary operation on morphisms[12], and so the morphisms actually form a monoid. This is the converse – if $\mathcal{C}$ is a category with only one object, we can define a monoid whose elements are the morphisms of $\mathcal{C}$, multiplication is composition, and the unit is the identity from the category.

Again, this correspondence may make you feal queasy at first, because a certain "level shift" has happened, so you don't know quite what you're standing on any more. It's worth thinking about it; when you start feeling comfortable with it you know you've made some progress in understanding something important in category theory.

4. We can generate a category by the diagram

where $f : x \longrightarrow x$ is *not* supposed to be the identity. So we also have non-trivial composites

$$\begin{array}{l} f \circ f \\ f \circ f \circ f \\ f \circ f \circ f \circ f \\ \vdots \end{array}$$

as well as an identity 1_x. Now this is a category with one object, so we know the morphisms form a monoid. Moreover, we know there's exactly one morphism for every natural number k – the composite $f \circ \cdots \circ f$ (k times). We do in fact get the additive monoid $\mathbb{N}$; note that 1_x corresponds to 0 ("compose f with itself no times"). We'll later see that this is the "free category on one object and one morphism" as well as being the free monoid on one element.

5. We could do something similar but impose the relation $f \circ f = 1$. This would turn our category into the one-object category corresponding to the cyclic group of order 2.

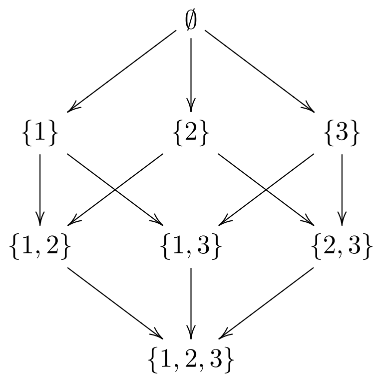
6. Every poset is a category. Given a poset P with ordering $\leq$, we make a category as follows:

   - the objects are the elements of P;

[12] Composition with source/target matching conditions is a "partial" binary operation – it isn't defined on *all* pairs of morphisms, only those pairs where the target of one is the source of the other.

- there is precisely one morphisms $x \longrightarrow y$ if $x \leq y$, and none otherwise. So a morphism $x \longrightarrow y$ is "the assertion $x \leq y$[13]" There is thus at most one morphism between any two objects. Composition is given by transitivity – given $x \leq y$ and $y \leq z$ we certainly have $x \leq z$, the "composite" of the first two assertions.

Here's an example – the power set of {1,2,3} is a poset ordered by inclusion. The objects are all possible subsets of {1,2,3}, and these organise themselves neatly as follows:



where the arrows show which sets are subsets of which others by only adding one element. However, there are also "composite" inclusions such as

$$\{1\} \subseteq \{1,2\} \subseteq \{1,2,3\}$$

and

$$\{1\} \subseteq \{1\}$$

which we haven't bothered drawing. And, lo and behold, this picture is exactly the diagram generating the category associated with this poset.

Note also, for example, that there is no composable path of arrows from $\{1,2\}$ to $\{2,3\}$ in the diagram – neither is a subset of the other. This is what makes this only a *partially* ordered set and not a totally ordered one.

7. There is a curious category **Mat** of matrices – with a warning that here the matrices are the *morphisms* not the objects. The objects are the natural numbers; the morphisms $n \longrightarrow m$ are all the $n \times m$ matrices (in some fixed field, k, say). Now let's think about what composition could possibly be. Given composable morphisms

$$n \xrightarrow{A} m \xrightarrow{B} l$$

[13] So we see that morphism doesn't have to be anything like a "map"; here it is an *assertion*, and this is not uncommon.

what this means is we have

- an $n \times m$ matrix A, and
- an $m \times l$ matrix B,

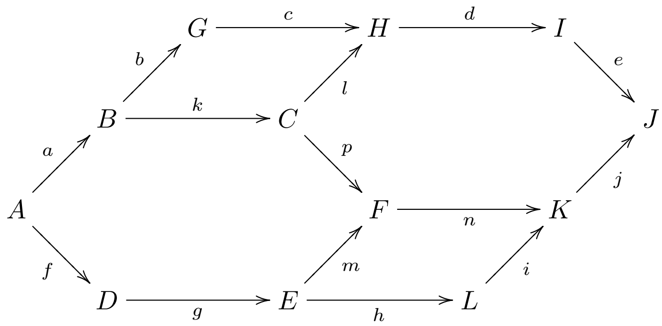
and we want to produce a "composite" $n \times l$ matrix – so we just multiply them and get

$$n \xrightarrow{AB} l.$$

You might notice that this is somehow related to the category of vector spaces and linear maps – at least, the finite-dimensional vector spaces, since we might then pick a basis and express the linear maps using matrices. We will make this relationship more precise later.

1.5 Exercises

1. Write out the string of equations of morphisms that this commutative diagram is expressing:



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2. Draw all the non-identity arrows in the category generated by the following diagram:

$$\cdot \longrightarrow \cdot \longrightarrow \cdot$$

Redraw the arrows so that they make a triangle. What algebraic object does this represent?

What about

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$$

and

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \ ?$$

What geometric shapes do these categories make when you draw in all the non-identity arrows? (Again, you'll want to redraw the arrows so that they're not all in a straight line.) What algebraic structures do they represent?

3. Show that in a poset (regarded as a category) all diagrams commute. Exhibit a square in **Set** that does not commute.

4. Let $x \xrightarrow{f} y$ be a morphism in a category. Guess what it means for f to be an isomorphism. Check it against what you think "isomorphism" should mean in the categories **Set**, **Grp**, **Mat**, **Top**.

5. Let A and B be sets, and $f : A \longrightarrow B$ an injection. Show that f has the following property: given $s, t : V \longrightarrow A$ such that $fs = ft$ then we must have $s = t$. *(Note: this is the definition of a "monomorphism" and in fact in* **Set** *this property completely characterises the injections.)*

6. A morphism $f : x \longrightarrow y$ is called a *split monomorphism* if there exists a morphism $s : y \longrightarrow x$ such that the composite

$$x \xrightarrow{f} y \xrightarrow{s} x$$

is the identity, and *split epimorphism* if there exists a morphism $p : y \longrightarrow x$ such that the composite

$$y \xrightarrow{p} x \xrightarrow{f} y$$

is the identity.

   i) What are the split monomorphisms in **Set**?
   ii) What are the split epimorphisms in **Set**? For this one, the axiom of choice makes a difference. Can you see why?
   iii) What is a morphism that is both a split monomorphism and a split epimorphism?

7. Guess the definition of "morphism of category" given the following sensible ideas:

   - A morphism of categories should respect the structure of a category, that is, composition and identities.
   - A one-object category is a monoid so a morphism of this should be a monoid homomorphism.