# Category Theory 2

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## Contents



## 2 Some basic universal properties

Many important concepts in Category Theory are determined or defined by a "universal property". In fact, one of the general principles of Category Theory is that universal properties are somehow an ideal way of describing "significatn structure". The idea of a universal property is to formalise our intuition about something being "canonical" – the best, the biggest or smallest, the most extreme, generic, free. Examples of things with universal properties include:

- cartesian products
- disjoint unions
- quotients
- equivalence classes
- free groups, free rings, etc
- completions

- closures
- abelianisation
- tensor products and direct sums of vector spaces
- direct and inverse limits
- simplicial sets
- geometric realisation
- nerves
- singular complex
- sheafification

One very important idea is that when we define something by a universal property it isn't exactly uniquely determined – it is "unique up to uniqu isomorphism". This means that any other object having this particular universal property will be isomorphic to the first one, via a *canonical* isomorphism. Here "canonical" will have a very precise meaning. This touches on another general principle of Category Theory – that our preferred notion of sameness is "canonically isomorphic". We'll see many examples of this; first of all we'd better define "isomorphism".

## 2.1 Isomorphisms

You might have already guessed what an isomorphism is.

**Definition 2.1.** A morphism $f : X \longrightarrow Y \in \mathcal{C}$ is an *isomorphism* if and only if there exists a morphism $g : Y \longrightarrow X$ such that $gf = 1_X$ and $fg = 1_y$. In this case $g$ is an *inverse* for $f$.

As you might expect, inverses are unique – if a morphism has an inverse at all, then it has only one. The proof of this is just the same as the proof that inverses are unique in a group – we suppose we have two inverses, use the fact that one of the maps is an inverse, and then the fact that the other is an inverse, and conclude that they must be the same.

**Proposition 2.2.** *If $g_1$ and $g_2$ are both inverses for $f : X \longrightarrow Y$ then $g_1 = g_2$.*

**Proof.**

$$\begin{array}{rcll} g_1 & = & g_1 \circ 1_Y & \text{by the definition of identities} \\ & = & g_1 \circ (f \circ g_2) & \text{since } g_2 \text{ is an inverse for } f \\ & = & (g_1 \circ f) \circ g_2 & \text{by associativity} \\ & = & 1_X \circ f & \text{since } g_1 \text{ is an inverse for } f \\ & = & g_2 & \text{by the definition of identities} \end{array}$$

□

**Proposition 2.3.**

*i) Identity maps are always isomorphisms.*

*ii) The composite of two isomorphisms is an isomorphism.*

**Proof.** Exercise for the reader. □

## 2.2 Terminal objects

We now come to our first universal property.

**Definition 2.4.** A *terminal object* in $\mathcal{C}$ is an object $T$ such that for any object $X \in \mathcal{C}$ there is a unique morphism $X \longrightarrow T$.

The part of that sentence after the words "such that" is the *universal property* that defines a terminal object.

***pic?

**Example: terminal objects in Set**

In the category of sets and functions, any 1-element set is terminal – there is a unique function from any other set because everything simply has to be mapped to the single element. For this reason we often call terminal objects 1 in any category.

Note that the set $\{0\}$ is terminal, but so are $\{7\}$, $\{64835\}$, and $\{x\}$. All these sets are isomorphic in a dull and straightforward way. Indeed, in any category, all the terminal objects are uniquely isomorphic. The proof is similar to the proof that inverses are unique – we assume we have two, and then we use the universal property of each in turn to show that they are in fact isomorphic.

**Proposition 2.5.** *Suppose $T$ and $T'$ are both terminal in $\mathcal{C}$. Then there exists a unique isomorphism $f : T \xrightarrow{\sim} T'$.*

**Proof.** We proceed in steps:

1. Since $T'$ is terminal, there is a unique morphism $f : T \longrightarrow T'$.
2. Since $T$ is terminal, there is a unique morphism $g : T' \longrightarrow T$.
3. Since $T$ is terminal, there is a unique morphism $T \longrightarrow T$ i.e. $1_T$.
4. The composite $g \circ f$ is a morphism $T \longrightarrow T$, so by (3) we must have $g \circ f = 1_T$.
5. Similarly, we must have $f \circ g = 1_{T'}$.

Hence $f$ is the unique isomorphism as required. □

### More examples

It is instructive to think about terminal objects in other categories and see whether or not they resemble 1-element sets at all:

- In **Gp**, the trivial group is terminal. Its only element is its unit, so it is indeed a 1-element set regarded as a group.
- In **Vect**, the zero vector space is terminal; this is rather similar to the case of groups.
- In **Top**, the point is terminal. This is a 1-element set regarded as a space.
- In a poset (regarded as a category as in ***ref), a terminal object is an element $t$ such that $x \leq t$ for all $x$ in the poset – so $t$ is the maximum (if there is one).
- The category of fields and field homomorphisms does not have a terminal object. We can't have the 1-element set regarded as a field, because a field must have at least two elements – 0 and 1 (we usually impose an axiom to ensure that $0 \neq 1$). Of course, this isn't really a proof that there isn't a terminal object, it's just pointing out that there isn't an obvious one. In fact, there really isn't one. ***why??

The last example showed us that not every category has a terminal object. Here are some very small categories with no terminal object:

- Non-example 1:

$$\cdot \rightrightarrows \cdot$$

- Non-example 2: any discrete category.
- Non-example 3:

$$\cdot \longrightarrow \cdot \qquad\qquad \cdot \longrightarrow \cdot$$

- Non-example 4: if we generate a category from a non-identity morphism $f : x \longrightarrow x$, we cannot have a terminal object.

$$\underset{x}{\overset{f}{\circlearrowright}}$$

## 2.3 Initial objects

**Definition 2.6.** An *initial object* in $\mathcal{C}$ is an object $I$ such that for any object $X \in \mathcal{C}$ there is a unique morphism $I \longrightarrow X$.

You may have noticed that there's something rather similar about the definitions of initial and terminal object, and you're right. We get one from the ohter by turning around the arrow in the definition. This is an important principle in Category Theory: the principle of duality. We'll discuss it formally later ***ref but here's the idea.

### The principle of duality

The idea is that we could consider all our arrows to be going in the opposite direction. It's quite obvious when you think about posets – we declared there to be a morphism $x \longrightarrow y$ whenever $x \leq y$ but that was slightly arbitrary. We could just as easily have said: whenever $x \geq y$. It's the same with any category – we could have declared all the morphisms $x \longrightarrow y$ to be morphisms $y \longrightarrow x$ instead.

Now, given any definition using morphisms, we can simply turn all the arrows around and get what is called the "dual" notion. Then any result we prove for the original notion is automatically proved in dual form as well – because we can turn all the arrows around in the proof as well! There is a completely precise, formal way of saying this which we'll come to a bit later.

We quite often (but not always) indicate that we've taken a dual by sticking the prefix "co" onto the name of the thing in question. Limits become colimits, monads become comonads, algebras become coalgebras. This does result in some slightly strange words though, such as cocone, cooperad, coring (how is that pronounced?), counit, countable. . .[1]

### Examples of initial objects

Here are some examples.

- In **Set**, the empty set $\emptyset$ is initial – there is a unique function from the empty set to any other set[2]. We often call initial objects 0, no matter what category they're in.
- In **Gp**, the trivial group is initial – and recall that it's also terminal. This is a special (and useful) sort of situation, and is very different from the case of sets, where the initial and terminal objects are very different.

[1] This is Richard Garner's joke, and you know you're getting into the spirit of Category Theory if you find it funny.

[2] If you have trouble with this idea, remember that a function $f : X \longrightarrow Y$ is defined by giving *for every element* $x \in X$ an element $f(x) \in Y$. If $X$ is empty, the "for every element in $X$" bit is vacuously satisfied. It's just like if someone gave you a box and said told you to run a mile for every elephant you find in the box. This task is rather easy if there are in fact no elephants in the box.

Other categories in which initial and terminal objects coincide include $\mathbf{Top}_*$ (based spaces) and $\mathbf{Set}_*$ (pointed sets). Remember that the empty set is *not* an object of $\mathbf{Set}_*$; the "smallest" possible pointed set is $\{*\}$, the set which has just one element, which is its basepoint. This has a unique morphism to any other pointed set, because the basepoint $*$ simply has to be mapped to the basepoint of the other pointed set, and there's nothing else to define.

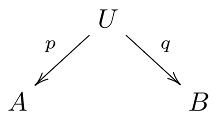
- In **Field** there is no initial object. ***proof?

In general initial objects can be thought of as the "smallest possible" thing with all the structure in question; this, like so many things, will be made precise later. In fact, the fact that **Field** has no initial object gives the result that we can't form a "free field" on a set. This is something very significant that comes of considering initial objects.

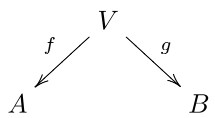
## 2.4 Products

Our next universal property is the notion of product.

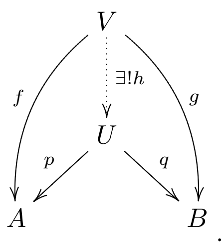
**Definition 2.7.** A *product* of objects $A$ and $B$ in $\mathcal{C}$ is an object $U$ equipped with morphisms as shown below




such that given any diagram

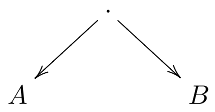



there exists a unique morphism $h$ making the following diagram commute[3]:




[3]The exclamation mark in this diagram is notation that means "unique", so "$\exists! h$" is to be read "there exists a unique $h$".

Here $U$ is for Universal. As before, the part of the definition after the words "such that" is giving the universal property in question. We can think of a product of $A$ and $B$ as being a universal diagram of the form




It is "universal among all such diagrams", the best possible such diagram. We also say that "every diagram of this form factors through it uniquely". The morphism $h$ is the "factor" in question, and the last diagram exhibits the factorisation – the bendy part of the diagram is a "product" (or rather, composite) of the universal diagram and the factor $h$. We also say that the morphism $h$ is "induced" by the universal property.

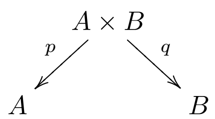
Note that although it's the object $U$ that is usually referred to as a product, it's really important that $U$ comes equipped with the maps $p$ and $q$, as we'll soon see.

**Example: products in Set**

This definition might seem a bit obscure to you at the moment, so let's work out what products are in our prototype category **Set**. You might be rather hoping that products in **Set** will turn out to be... products, that is, cartesian products. And you're right! Phew. But can we actually show that this is true? To do that, we have to take two sets $A$ and $B$ and show that the cartesian product $A \times B$ has the required universal property. Recall that[4]

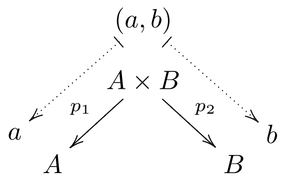
$$A \times B = \{\ (a, b) \mid a \in A, b \in B\}.$$

Do we have functions

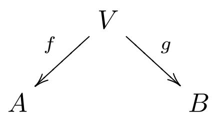



that are somehow "obvious" enough to give a universal property? The answer is yes: we have the "projection" maps that send the pair $(a, b)$ to $a$ on the one hand, and to $b$ on the other, as shown below:

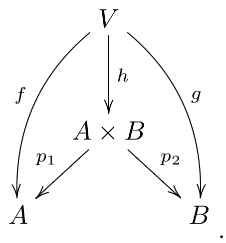
[4]The vertical line | is to be read "such that", so this notation is to be read "the set of all pairs $(a, b)$ such that $a$ is an element of $A$ and $b$ is an element of $B$."




We call these $p_1$ for "project onto the first component" and $p_2$ for "project onto the second component". Now, does this have the right universal property? Given functions




can we define a function $h : V \longrightarrow A \times B$ making the correct diagram commute? Well, let's consider an element $v \in V$, and suppose we have a function $h$ with $h(v) = (a, b)$. Now in order for the diagram




to commute, we must have

$$\begin{array}{rcll} f(v) & = & (p_1 \circ h)(v) & \\ & = & p_1(h(v)) & \text{by the definition of composition of functions} \\ & = & p_1(a, b) & \text{by the notation we introduced above} \\ & = & a & \text{by the definition of } p_1 \end{array}$$

and similarly

$$\begin{array}{rcl} g(v) & = & (p_2 \circ h)(v) \\ & = & p_2(h(v)) \\ & = & p_2(a, b) \\ & = & b \end{array}$$

So we define a function $h : V \longrightarrow A \times B$ by

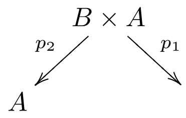
$$h(v) = (f(v), g(v))$$

and it is indeed the unique function making the required diagram commute.

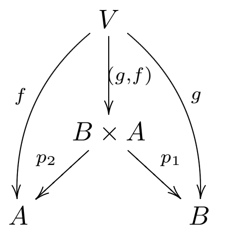
Incidentally, inspired by this example, we write this product as $A \times B$ and the induced factor as $(f, g)$, and we generally refer to the maps $p$ and $q$ as "projections".

### Uniqueness of products

Are products unique? You can probably guess that the answer is: they are unique up to unique isomorphism. Let's think about this in **Set** for a second. If we were feeling a bit perverse, we might decide that the product of $A$ and $B$ should be $B \times A$. Would this work? The diagram exhibiting this product would then be:




and the factorisation would be




Of course, $B \times A$ and $A \times B$ are isomorphic in a very very canonical way. There are even more perverse ways of forming specific products in **Set**. For example, given

$$A = \{0, 1\}$$
$$B = \{0, 1\}$$

the most obvious product is

$$A \times B = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

but the set

$$\{1, 2, 3, 4\}$$

can also be given the structure of a (categorical) product of $A$ and $B$. All we have to do is define the projects $p$ and $q$. For example, we could define $p$ by

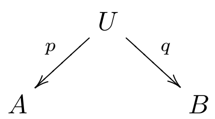
$$\begin{array}{ccc} 1 & \mapsto & 0 \\ 2 & \mapsto & 0 \\ 3 & \mapsto & 1 \\ 4 & \mapsto & 1 \end{array}$$

and $q$ by

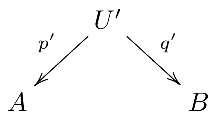
$$\begin{array}{ccc} 1 & \mapsto & 0 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 0 \\ 4 & \mapsto & 1 \end{array}$$

If you think about why this gives a product, you will (hopefully) see that all we've really done here is "imagine" that the elements 1,2,3,4 are actually the elements $(0,0),(1,0),(0,1),(1,1)$ in disguise, and we've defined the projection maps accordingly. In fact, defining maps corresponds precisely to *telling the secret* of which element is disguised as which. It should now be clear that *any* 4-element set could be given the structure of $A \times B$ in this case. Moreover, once the projection maps are defined, these products all become uniquely isomorphic. In general, 4-element sets are all isomorphic but in many possible ways; however, here we are only interested in isomorphisms that *respect the product structure*, that is, that maps $(0,0)$-in-disguise in one product, to $(0,0)$-in-disguise in the other, and likewise for all the other elements. Before this analogy goes much too far (if it hasn't already), we'd better make that precise.

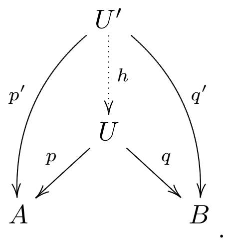
**Proposition 2.8.** *Given products*




*and*

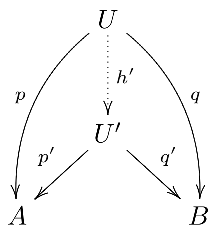



*there exists a unique isomorphism* $h : U \stackrel{\sim}{\longrightarrow} U'$ *making the following diagram commute*

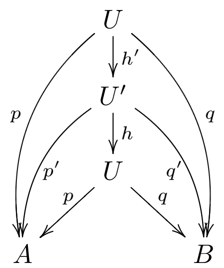



Note that there may be many isomorphisms $U \longrightarrow U'$, but only one *making the diagram commute.*

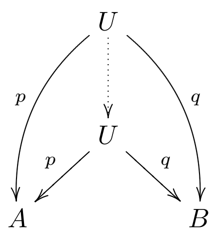
**Proof.** The universal property of $U$ induces a unique morphism $h$ making the above diagram commute, so it only remains to show that $h$ is an isomorphism. The universal property of $U'$ induces a unique morphism $h' : U' \longrightarrow U$ making the following diagram commute:




and we can now show that $h'$ is inverse to $h$. The following diagram commutes:




which means in particular that the morphism $h \circ h' : U \longrightarrow U$ makes the following diagram commute




but $1_U : U \longrightarrow U$ also makes this diagram commute. By the universal property of $U$ there is a unique morphism $U \longrightarrow U$ making this diagram commute, and so we must have $h \circ h' = 1_U$.

Similarly, we must have $h' \circ h = 1_{U'}$, so $h'$ is indeed inverse to $h$, and $h$ is the unique isomorphism required. □

You may notice that this proof followed the exact same structure as our proof of the analogous uniqueness result for terminal objects. This is no coincidence! Terminal objects and products are both examples of *limits*, and there is a general proof that limits are unique up to unique isomorphism in this very way; we have now seen two specific examples of this. It is useful to warm up to the general case using these small examples, because when we finally see the formal general definition of limit it will help to have some intuition about what it's supposed to be saying.

Here's another useful result about products, that formalises something we know already about cartesian products of sets. Given sets $A, A', B, B'$ and functions $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$, we get a function

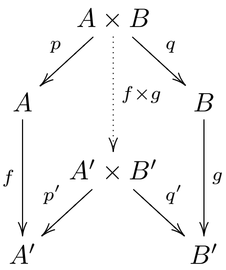
$$f \times g : A \times B \longrightarrow A' \times B'$$

defined simply by

$$(a, b) \mapsto (f(a), g(b)).$$

We will now see that although we have defined this on elements[5], it in fact follows *just from the categorical definition of product.*

**Proposition 2.9.** *Given products $A \times B$ and $A' \times B'$ and morphisms $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$, there is a unique morphism $f \times g$ making the following diagram commute:*




**Proof.** The required morphism is immediately induced from the universal property of $A' \times B'$. □

Note that we didn't use an awful lot of the product structure of $A \times B$; the point is that the induced factorisation only *deserves* to be called $f \times g$ if it's going from a product to a product in this way. You may wish to check that in **Set** the morphism induced in this way really is the one we first thought of above.

### More examples of products

- In **Gp**, products are direct products, that is, we take the cartesian product of the underlying sets and induce the obvious pointwise group operation. In case it's not "obvious" to you: given groups $A$ and $B$ we define a group operation on the cartesian product of their underlying sets by

$$(a, b) \circ (a', b') = (a \circ a', b \circ b').$$

[5]To turn a fact about sets and functions into a fact about objects an morphisms in *any* category, we have to express the fact without ever referring to elements of sets.

Of course, anything isomorphic to this will also do.

- In **Top**, products are product spaces; again the underlying set is the cartesian product of the underlying sets. And again, anything isomorphic to this will also do.
- In **Vect**, products are direct products. And yet again, the underlying set of is the cartesian product of the underlying sets, anything isomorphic to this will also do.

Aside: the property that "the underlying set of the product is the product of the underlying sets" is a useful one, and will later be honoured with a generalisation, and a name.

- In an ordered set (regarded as a category), the product of $x$ and $y$ is the minimum of $x$ and $y$. It is actually unique. To see that this is true, suppose $x \leq y$, so the claim is that $x \times y = x$. We must check it has the required universal property, which translates into the ordered set as: given $v$ such that $v \leq x$ and $v \leq y$, then $v \leq x$. This is clearly true!
- In a poset (regarded as a category), the product of $x$ and $y$ is the greatest lower bound, or meet, of $x$ and $y$. (If you don't know what a meet is, you now do: it's the categorical product of $x$ and $y$!)
- Here's an oh-so-witty example. Let $\mathcal{C}$ be the category whose objects are the natural numbers. For morphisms: there is precisely one morphism $n \longrightarrow m$ if $n$ divides $m$, and none otherwise. This is in fact a poset. Anyway, in this example we get to write silly things like $6 \times 8 = 2$, and $24 \times 60 = 12$, and everyone laughs. Seriously though – the product of $n$ and $m$ in this category is their highest common factor, which is really quite satisfying.

### Categories with products

In general we say that a category "has (all) binary products" if any two objects have a product. There is a more general notion of products of more than two objects which we'll come to later.

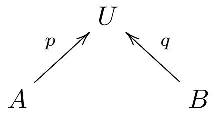
Thus, **Set**, **Gp**, **Top** and **Vect** have binary products. An ordered set regarded as a category has binary products. A poset regarded as a category might not have binary products; it depends on the poset.

## 2.5 Coproducts

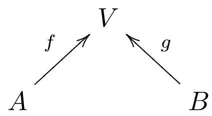
Coproducts are the dual of products, as the "co" prefix indicates. In a way there's nothing more to say – if you understand the *principle of duality* you'll be able to write down the definition immediately. However, we will now spell it out, for the record[6].

[6] And because it's pretty easy to just turn all the arrows around in the LaTeXcode! ***xyPic

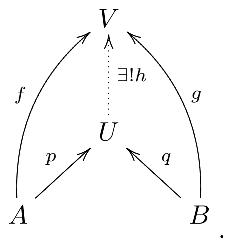
**Definition 2.10.** A *coproduct* of objects $A$ and $B$ in $\mathcal{C}$ is an object $U$ equipped with morphisms as shown below




such that given any diagram




there exists a unique morphism $h$ making the following diagram commute:

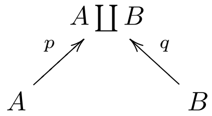



### Reassurance about directions

You might be wondering how you're ever going to remember which direction $h$ goes in products as opposed to coproducts. Now, there are a lot of times in Category Theory when it really is hard to remember which way round things go[7] but this really isn't one of them. You just have to remember that the universal property of your universal thingy says that any other thingy *factors through* it. Then as long as you understand what a factor is, you can't get your directions wrong.

### Examples of coproducts

- In **Set**, coproducts are disjoint unions. The maps $p$ and $q$ are the *insertions*




[7] Distributive laws are my current favourite example of this.

and we often use this terminology for the analogous morphisms for coproducts in other categories as well; we also sometimes say "coprojections" since they are the dual of projections.

- In **Top**, coproducts are also disjoint unions.
- In **Gp**, coproducts are "free products", sometimes written $A * B$. You basically take all the elements of $A$ and $B$ and then generate a group freely while preserving the group operations of $A$ and $B$ (so the free part is when you're multiplying an element of $a$ with an element of $b$). Defining this rigorously is quite hard, and indeed it is often the case in algebraic structures that the products are obvious and the coproducts are much less obvious[8]. However, the idea is clear – take the disjoint union of the underlying sets and then form the smallest possible structure of the kind you want. For spaces the disjoint union was *already* a structure of the kind we wanted. For groups, it wasn't.
- In **Vect** the situation is rather similar to the case for groups; after all vector spaces are special kinds of groups. Anyway here, coproducts are *direct sums*. This conflation of sum/product/coproduct terminology is not very helpful, I'm sure you'll agree.
- In $\mathbf{Set}_*$, we make coproducts by basically taking the disjoint union of two sets, but we then identify their basepoints to make the new basepoint. This is also the case in $\mathbf{Top}_*$ but it gets the fancy name "wedge". Hence in topology you hear people talking about "the wedge of two circles" or "a wedge of spheres".

## 2.6 Pullbacks

Here's a slightly more complicated sort of universal property.

**Definition 2.11.** A *pullback square* is a commutative square

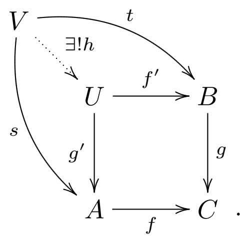
$$\begin{array}{ccc} U & \xrightarrow{g'} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

such that given any commutative square

$$\begin{array}{ccc} V & \xrightarrow{t} & C \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

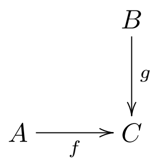
[8] This actually has a precise formulation in terms of "creation of limits".

there is a unique factorisation $h$ making the following diagram commute

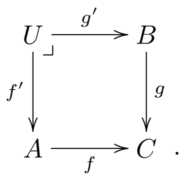



In this case we also say that $g'$ is a pullback of $g$ over (or along) $f$, or equivalently that $f'$ is a pullback of $f$ over (or along) $g$.

You will notice that we've called the top left corner "$U$" for Universal. This is because it's at that corner that all the factoring happens, and it's the object at that corner that is the new one – we might well start with the data




and produce the pullback square from it. So we sometimes even say "$U$ is a pullback" if we're feeling particularly sloppy. To indicate a pullback square, we often put a little $\lrcorner$ sign in the corner like this:




Pullbacks are of course unique up to unique isomorphism; that is, given any two pullbacks of the same morphisms, there exists a unique isomorphisms between them making the relevant diagram commute.

### Pullbacks in Set

We can construct a pullback in **Set** by taking $U$ to be the following subset of $A \times B$:

$$\{\ (a, b) \in A \times B \mid f(a) = g(b)\ \}$$

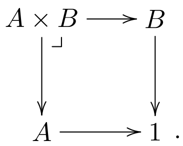
and then $g'$ and $f'$ are the restricted projections onto the first and second component respectively. For this reason we sometimes write a pullback as

$$A \times_C B$$

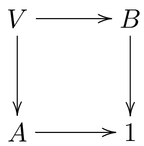
which is fine if $f$ and $g$ are obvious and canonical, but severely insufficient if they're not. Pullbacks are also called also called a *fibred product* or *cartesian square*.

### Products as pullbacks

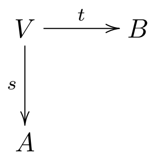
If $\mathcal{C}$ has a terminal object, then a product $A \times B$ can be expressed as a pullback:



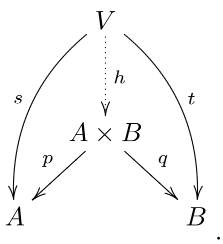
We check that this has the correct universal property. Now to give a commutative square



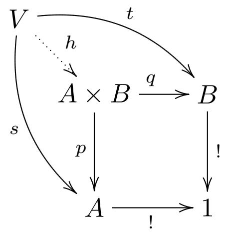
we just have to give a pair of morphisms



because the other two morphisms are uniquely determined by the fact that 1 is terminal; furthermore because 1 is terminal the square must commute (since there is a unique morphisms $V \longrightarrow 1$). Now the universal property of the product $A \times B$ induces a unique morphism $h$ making the following diagram commute



But this diagram commutes if and only if the following diagram commutes

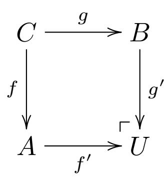



and so the universal property is the correct one.

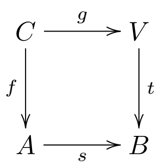
## 2.7 Pushouts

Pushouts are dual to pullbacks. Again, we'll spell it out although it is usual in this case not just to reverse the arrows but also to turn the diagram around on the page.

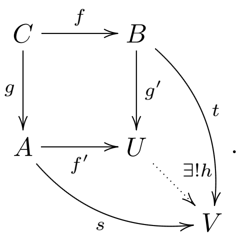
**Definition 2.12.** A *pullback square* is a commutative square




such that given any commutative square




there is a unique factorisation $h$ making the following diagram commute




In this case we also say that $g'$ is a pushout of $g$ along $f$, or equivalently that $f'$ is a pushout of $f$ along $g$.

### Unions as pushouts

We know that pullbacks are somehow related to pushouts, so dually we expect pushouts to be somehow related to coproducts. In **Set** this can manifest itself in the form of *unions* – as opposed to disjoint unions i.e. coproducts. The following is a pushout square in **Set**, where the morphisms are the obvious inclusions:

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \cup B \end{array} .$$

There are more general pushouts than this as well. ***say?

## 2.8 Equalisers

**Definition 2.13.** Given morphisms

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

an *equaliser* for them is a "fork" that is morphism $e$ as below with $fe = ge$

$$U \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

such that given any fork

$$V \xrightarrow{s} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

there is a unique factorisation

$$\begin{array}{ccccc} U & \xrightarrow{e} & A & \underset{g}{\overset{f}{\rightrightarrows}} & B \\ {\scriptstyle h}\uparrow & \nearrow_{s} & & & \\ V & & & & \end{array} .$$

### Example: equalisers in Set

In **Set** we get the subset of $A$ defined by:

$$\{ \ a \in A \mid f(a) = g(a) \ \};$$

the map $e$ is then just the inclusion of this subset into $A$. Note that we would still have a fork if we threw in some more elements to this subset, as long as $e$ mapped them all to elements in the original subset of $A$. However, this would no longer be a *universal* fork – we would have existence of factorisations, but not uniqueness. On the other hand if we removed some elements of this subset, we might not have existence any more, but if a factorisation did exist it would be unique.

## 2.9 Coequalisers

Coequalisers are the dual of equalisers.

**Definition 2.14.** Given morphisms

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

a *coequaliser* for them is a fork

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{c} U$$

that is, $cf = cg$, such that given any fork

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{s} V$$

there exists a unique factorisation.

$$\begin{array}{ccccc} A & \underset{g}{\overset{f}{\rightrightarrows}} & B & \xrightarrow{e} & U \\ & & & \searrow^{s} & \vdots\, h \\ & & & & V \end{array}$$

**Examples**

Coequalisers in **Set** are quotients. With the above notation we get $B/\sim$ where we are quotienting out by the smallest equivalence relation generated by $f(a) \sim g(a)$ for all $a \in A$.

The slogan "coequalisers are quotients" isn't far wrong. Coequalisers give quotient spaces in **Top** and quotient groups in **Gp**[9]. In fact, it's using coequalisers that we generalise the fact that "every group is a quotient of a free group". Coequalisers seem to crop up more than equalisers, even though they're the ones with the "co" in front of them.

[9]Technically we can only quotient out by a normal subgroup, so for general groups we'll end up quotienting out by a normal subgroup generated by the non-normal subgroup in question.

## 2.10 Exercises

1. Show that for any object $X$ in a category $\mathcal{C}$, $1_X$ is an isomorphism.

2. Show that if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are isomorphisms then $g \circ f$ is an isomorphism.

3. What is an isomorphism in **Rel**, the category of sets and relations?

4. Let $\mathcal{C}$ be the category as in the oh-so-witty example. That is, objects are natural numbers and there is precisely one morphism $n \longrightarrow m$ if $n$ divides $m$, and none otherwise. What are coproducts in this category?

5. Write down the relevant uniqueness property for pushouts, and prove that it holds.

6. Guess the definition of 3-fold product, that is, product of three objects $A, B, C$. Show that in **Set** the cartesian product $A \times B \times C$ is a 3-fold product, as are $(A \times B) \times C$ and $A \times (B \times C)$.

7. Show how to express a coproduct as a pushout. Hint: just dualise the whole argument for expressing a product as a pullback.

8. In **Set** there is a useful diagonal map

$$\begin{array}{cccc} \Delta : & A & \longrightarrow & A \times A \\ & a & \mapsto & (a, a) \end{array}$$

   Show how to induce this using the universal property of the product $A \times A$, thus giving us a notion of "diagonal" in any category with products. What is the dual notion?

9. Let $f, g : A \longrightarrow B$. Show that the following pushout provides an equaliser for $f$ and $g$:

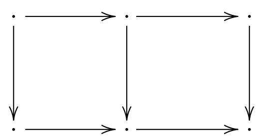
$$\begin{array}{ccc} \cdot & \longrightarrow & B \\ \downarrow & & \downarrow \Delta \\ A & \xrightarrow[(f,g)]{} & B \times B \, . \end{array}$$

   Here $\Delta : B \longrightarrow B \times B$ is the "diagonal" function defined above, that maps $b$ to $(b, b)$.

10. i) What are products in $\mathbf{Set}_*$?

    ii) What are products in $\mathbf{Top}_*$?

11. Suppose that



is a commutative diagram.

i) Show that if both small squares are pullbacks then so is the large rectangle.

ii) Show that if the large rectangle and the right hand square are pullbacks, then so is the left hand square.

iii) Deduce from the above (or prove directly) that the pullback of a pullback square is a pullback square, stating clearly what you take this to mean.