CATEGORY THEORY

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1 · Categories, functors and natural transformations

1.1 · *Categories*

DEFINITION 1.1.1

A *category* $\mathcal{C}$ consists of:

- a collection of objects, $\mathrm{ob}\,\mathcal{C}$;
- For every pair $X, Y \in \mathrm{ob}\,\mathcal{C}$, a collection $\mathcal{C}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms $f\colon X \longrightarrow Y$,

equipped with:

- for each $X \in \mathrm{ob}\,\mathcal{C}$, an identity map $\mathrm{id}_X = 1_X \in \mathcal{C}(X, X)$;
- for each $X, Y, Z \in \mathrm{ob}\,\mathcal{C}$, a composition map

$$\begin{aligned} m_{XYZ}\colon \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) &\longrightarrow \mathcal{C}(X,Z) \\ (g, f) &\longmapsto g \circ f = gf, \end{aligned}$$

satisfying:

- unit laws — if $f\colon X \longrightarrow Y$ then $1_Y \circ f = f = f \circ 1_X$
- associativity — if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, then $h(gf) = (hg)f$.

A category is said to be *small* if $\mathrm{ob}\,\mathcal{C}$ and all of the $\mathcal{C}(X, Y)$ are sets, and *locally small* if each $\mathcal{C}(X, Y)$ is a set.

REMARKS

1. If $f \in \mathcal{C}(X, Y)$, we say that X and Y are the *domain* (or *source*) and the *codomain* (or *target*) of f.
2. Morphisms are also referred to as *maps* or *arrows*.
3. We can write $\mathrm{Hom}_{\mathcal{C}}$ for the collection of all morphisms.
4. It is convenient and customary to assume that the $\mathcal{C}(X, Y)$ are disjoint for distinct pairs (X, Y).
5. We don't worry ourselves with the niceties of set theory.

DEFINITION 1.1.2

A category $\mathcal{C}$ is called *discrete* if the only morphisms are identities; i.e.

$$\mathcal{C}(X, Y) = \begin{cases} \{1_X\} & \text{if } X = Y \\ \varnothing & \text{otherwise.} \end{cases}$$

EXAMPLES 1.1.3

1. Large categories of mathematical structures:
   a. **Set** of sets and functions.
   b. Categories derived from or related to **Set**:

- **Pfn** of sets and partial functions;
- **Rel** of sets and relations;
- **Set**$_*$ of pointed sets and base point preserving functions.

c Algebraic structures and structure-preserving maps:

- **Grp** of groups and group homomorphisms;
- **Ab** of abelian groups and group homomorphisms;
- **Ring** of rings and ring homomorphisms;
- **Vec** of vector spaces over $\mathbb{R}$;
- **Mat** of natural numbers and $n \times m$ matrices.

d Topological categories:

- **Top** of topological spaces and continuous maps;
- **Haus** of Hausdorff spaces and continuous maps;
- **Met** of metric spaces and uniformly continuous maps;
- **Htpy** of topological spaces and homotopy classes of maps.

2 Mathematical structures as categories:

a Posets: a poset $(P, \leqslant)$ can be regarded as a category $\mathcal{C}$ with objects the elements of P and precisely one morphism $x \longrightarrow y$ when $x \leqslant y$ and none otherwise.
b Monoids: a category with just one object is a monoid.
c Groups: a group G can be regarded as a category with just one (formal) object and whose morphisms are the elements of G.

3 Small categories can be presented by generators and relations. From a directed graph we can generate a category of "paths through the graph" and then add relations imposing equalities between some paths with the same domain and codomain.

a There is a category **0** with no objects and no morphisms, generated by the empty graph.
b There is a category **1** with one objects and one (identity) morphism, generated by the graph with just one vertex.
c There is a category generated by the graph with one vertex and one edge. It is isomorphic to the additive monoid $\mathbb{N}$.
d There is a category generated by the graph with one vertex and one edge s say, together with the relation $s^2 = 1$. It has one object and two morphisms and is isomorphic to the cyclic group of order 2.
e There is a category generated by the graph with two vertices and one edge between them. It has two objects and three morphisms and is isomorphic to the poset $\mathbf{2} = \{0 \leqslant 1\}$.

1.2 · *Universal properties*

DEFINITION 1.2.1

A morphism $f \in \mathcal{C}(X, Y)$ is an *isomorphism* if $\exists g \in \mathcal{C}(Y, X)$ such that $gf = 1_X$ and $fg = 1_Y$. We say g is an *inverse* for f.

PROPOSITION 1.2.2

If g_1 and g_2 are inverses for f, then $g_1 = g_2$.

PROOF

$$g_1 = g_1 \circ 1_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = 1_X \circ g_2 = g_2. \qquad \square$$

PROPOSITION 1.2.3

1 The identity map is an isomorphism.
2 The composition of two isomorphisms is an isomorphism.

PROOF

1 1_X is clearly self-inverse.
2 Let $f \in \mathcal{C}(Y, Z)$, $g \in \mathcal{C}(X, Y)$ be isomorphisms, with respective inverses $h \in \mathcal{C}(Z, Y)$, $k \in \mathcal{C}(Y, X)$. Then we claim that $fg \in \mathcal{C}(X, Z)$ is an isomorphism, with inverse $kh \in \mathcal{C}(Z, X)$. For

$$(fg)(kh) = f(gk)h = f(1_Y)h = fh = 1_Z$$
$$(kh)(fg) = k(hf)g = k(1_Y)g = kg = 1_X$$

so we have the desired result. □

DEFINITION 1.2.4

A *terminal object* in $\mathcal{C}$ is an element $T \in \operatorname{ob}\mathcal{C}$ such that $\forall X \in \mathcal{C}$, $\exists!$ morphism $X \xrightarrow{k} T$.

EXAMPLE

In **Set**, every 1-element set is terminal. So sometimes we denote a terminal object by 1.

PROPOSITION 1.2.5

Suppose 1 and $1'$ are terminal in $\mathcal{C}$. Then there exists a unique isomorphism $f \in \mathcal{C}(1, 1')$.

PROOF

Since $1'$ is terminal, there is a unique morphism $f\colon 1 \longrightarrow 1'$. Similarly, 1 is terminal, so there is a unique morphism $f'\colon 1' \longrightarrow 1$. Now consider $f' \circ f \in \mathcal{C}(1, 1)$. Since 1 is terminal, there is a unique morphism $1 \longrightarrow 1$, i.e. the identity. So $f' \circ f = \mathrm{id}_1$; similarly $f \circ f' = \mathrm{id}_{1'}$. Hence f is the desired unique isomorphism. □

DEFINITION 1.2.6

Given $A, B \in \operatorname{ob}\mathcal{C}$, a *product* of A and B is an object $A \times B$ equipped with projections

$$\begin{tikzcd} & A \times B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & B, \end{tikzcd}$$

such that for all $f\colon C \longrightarrow A$, $g\colon C \longrightarrow B$, $\exists!$ morphism $(f, g)\colon C \longrightarrow A \times B$ such that $p \circ (f, g) = f$ and $q \circ (f, g) = g$; i.e. such that

$$\begin{tikzcd} & C \arrow[d, "{(f,g)}"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & A \times B \arrow[dl, "p"] \arrow[dr, "q"'] & \\ A & & B \end{tikzcd}$$

commutes.

EXAMPLE

In **Set**, $A \times B = \{ (a, b) \mid a \in A, b \in B \}$ with p, q the first and second projections.

Note however, that we could also have taken p, q to be the second and first projections, or the set to be $\{ (b, a) \mid b \in B, a \in A \}$.

PROPOSITION 1.2.7

If

$$\begin{tikzcd} & D \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & D' \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\ A & & B \end{tikzcd}$$

are products of $A, B \in \mathcal{C}$, then $\exists!$ isomorphism $k\colon D \longrightarrow D'$ such that $q'k = q$ and $p'k = p$.

PROOF

Consider the diagrams

$$\begin{tikzcd} & D \arrow[d, "k"] \arrow[ddl, bend right, "p"'] \arrow[ddr, bend left, "q"] & \\ & D' \arrow[dl, "p'"] \arrow[dr, "q'"'] & \\ A & & B \end{tikzcd} \quad \begin{tikzcd} & D' \arrow[d, "k'"] \arrow[ddl, bend right, "p'"'] \arrow[ddr, bend left, "q'"] & \\ & D \arrow[dl, "p"] \arrow[dr, "q"'] & \\ A & & B. \end{tikzcd}$$

By our definition of product, k is the unique morphism $D \longrightarrow D'$ s.t. these diagrams commute; so $q'k = q$ and $p'k = p$ certainly.

We claim that k' is an inverse for k. For consider $k \circ k'\colon D' \longrightarrow D'$. We have

$$p' \circ (k \circ k') = (p' \circ k) \circ k' = p \circ k' = p'$$
$$q' \circ (k \circ k') = (q' \circ k) \circ k' = q \circ k' = q'$$

Hence

$$\begin{tikzcd} & D' \arrow[d, "k \circ k'"] \arrow[ddl, bend right, "p'"'] \arrow[ddr, bend left, "q'"] & \\ & D' \arrow[dl, "p'"] \arrow[dr, "q'"'] & \\ A & & B \end{tikzcd}$$

commutes. But by the definition of product, there is a unique morphism $D' \longrightarrow D'$ that makes this diagram commute, i.e. the identity. So $k \circ k' = \mathrm{id}_{D'}$. Similarly $k' \circ k = \mathrm{id}_D$. So k is indeed an isomorphism, and is the unique one s.t. $q'k = q$ and $p'k = p$. □

DEFINITION 1.2.8

If $\forall A, B \in \mathcal{C}$, there exists a product $A \times B$, we say $\mathcal{C}$ has all binary products.

PROPOSITION 1.2.9

If $\mathcal{C}$ is a category with binary products, then given $f \in \mathcal{C}(A, C)$, $g \in \mathcal{C}(B, D)$, there exists a unique morphism $f \times g \in \mathcal{C}(A \times B, C \times D)$ such that

$$
\begin{tikzcd}
& A \times B \arrow[dl, "p_0"'] \arrow[dr, "q_0"] \arrow[dd, "f\times g"] & \\
A \arrow[dd, "f"'] & & B \arrow[dd, "g"] \\
& C \times D \arrow[dl, "p_1"] \arrow[dr, "q_1"'] & \\
C & & D
\end{tikzcd}
$$

commutes.

PROOF

Immediate from definition of product. □

LECTURE 3 · 16/10/02

DEFINITION 1.2.10

Suppose $\mathcal{C}$ is a category with binary products. Given $B, C \in \operatorname{ob}\mathcal{C}$, a *function space* or *exponential* is an object C^B equipped with an *evaluation morphism* $\varepsilon\colon C^B \times B \longrightarrow C$ such that $\forall f\colon A \times B \longrightarrow C$, $\exists!\, \overline{f}\colon A \longrightarrow C^B$ such that

$$
\begin{tikzcd}
A \times B \arrow[rr, "f"] \arrow[dr, "\overline{f}\times 1_B"'] & & C \\
& C_B \times B \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
$$

commutes, i.e. $\varepsilon \circ (\overline{f} \times 1_B) = f$.

In **Set**, $C^B = \{f\colon B \longrightarrow C\} = [B, C]$. There is an evaluation map

$$
\begin{aligned}
\varepsilon\colon C^B \times B &\longrightarrow C \\
(g, b) &\longmapsto g(b).
\end{aligned}
$$

Given $f\colon A \times B \longrightarrow C$, fix $a \in A$ to get

$$
\begin{aligned}
f_a\colon B &\longrightarrow C \\
b &\longmapsto f(a, b).
\end{aligned}
$$

So we have a function

$$
\begin{aligned}
\overline{f}\colon A &\longrightarrow C^B \\
a &\longmapsto f_a,
\end{aligned}
$$

such that

$$
\begin{aligned}
f(a, b) &= f_a(b) \\
&= \varepsilon(f_a, b) \\
&= \varepsilon \circ (\overline{f} \times 1_B)(a, b).
\end{aligned}
$$

So $\varepsilon \circ (\overline{f} \times 1_B) = f$ as required.

DEFINITION 1.3.1

A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of subcollections

- $\mathrm{ob}\,\mathcal{D} \subseteq \mathrm{ob}\,\mathcal{C}$;
- $\mathrm{Hom}_{\mathcal{D}} \subseteq \mathrm{Hom}_{\mathcal{C}}$,

together with composition and identities inherited from $\mathcal{C}$. We say $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$ if $\forall X, Y \in \mathcal{D}$, $\mathcal{D}(X, Y) = \mathcal{C}(X, Y)$, and a *lluf subcategory* of $\mathcal{C}$ if $\mathrm{ob}\,\mathcal{C} = \mathrm{ob}\,\mathcal{D}$.

We can think of the data for a category as

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}} \arrow[r, bend left, "c_1"] \arrow[r, bend right, "c_2"'] & \mathrm{ob}\,\mathcal{C} \end{tikzcd}$$

We could have c_1 giving us the domain of a morphism and c_2 the codomain, or vice verse. This motivates the definition:

DEFINITION 1.3.2

Given a category $\mathcal{C}$, the *dual* or *opposite* category $\mathcal{C}^{\mathrm{op}}$ is defined by:–

- $\mathrm{ob}\,\mathcal{C} = \mathrm{ob}\,\mathcal{C}^{\mathrm{op}}$;
- $\mathcal{C}(X, Y) = \mathcal{C}^{\mathrm{op}}(Y, X)$;
- identities inherited;
- $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$.

THE PRINCIPLE OF DUALITY

Given any property, feature or theorem in terms of diagrams of morphisms, we can immediately obtain its dual by reversing all the arrows (this is often indicated by the prefix "co-").

EXAMPLES 1.3.3

1. The dual notion of a terminal category object is an *initial* object. That is, an object $I \in \mathcal{C}$ such that for all $Y \in \mathcal{C}$, there exists a unique $f: I \longrightarrow Y$. For example, the (unique) initial object in **Set** is $\varnothing$; we sometimes write 0 for an initial object.
2. The dual of a product is a *coproduct*:

$$\begin{tikzcd} & A \amalg B & \\ A \arrow[ur, "p"] & & B \arrow[ul, "q"'] \end{tikzcd}$$

where p, q are *coprojections* such that, for any $f \in \mathcal{C}(A, C)$, $g \in \mathcal{C}(B, C)$, $\exists!\, h: A \amalg B \longrightarrow C$ such that

$$\begin{tikzcd} & C & \\ & A \amalg B \arrow[u, "h"'] & \\ A \arrow[uur, bend left, "f"] \arrow[ur, "p"'] & & B \arrow[uul, bend right, "g"'] \arrow[ul, "q"] \end{tikzcd}$$

commutes.

DEFINITION 1.3.4

A morphism $A \xrightarrow{m} B$ is *monic* iff given any $f, g: C \longrightarrow A$, we have $mf = mg \Rightarrow f = g$. Dually, a morphism $A \xrightarrow{e} B$ is *epic* iff given any $f, g: B \longrightarrow C$, we have $fe = ge \Rightarrow f = g$.

It is easy to see that any isomorphism is epic and monic. In **Set**, a morphism is monic iff it is injective, and epic iff it is surjective.

DEFINITION 1.3.5

Given $\mathcal{C}$ a category and $X \in \operatorname{ob}\mathcal{C}$, then the *slice over* X, $\mathcal{C}/X$ is the category with:

- objects (Y, f), where $f\colon Y \to X \in \mathcal{C}$;
- morphisms $h\colon (Y_1, f_1) \to (Y_2, f_2)$ such that

$$\begin{tikzcd} Y_1 \arrow[rr, "h"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

commutes, i.e. $f_2 h = f_1$.

Dually, we have the *slice under* X, $X/\mathcal{C}$, with:

- objects (Y, f), where $f\colon X \to Y \in \mathcal{C}$;
- morphisms $h\colon (Y_1, f_1) \to (Y_2, f_2)$ such that

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ Y_1 \arrow[rr, "h"'] & & Y_2 \end{tikzcd}$$

commutes, i.e. $hf_1 = f_2$.

We have a terminal object $(X, 1_X)$ in $\mathcal{C}/X$ and dually an initial object $(X, 1_X)$ in $X/\mathcal{C}$.

1.4 · *Functors*

DEFINITION 1.4.1

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ associates

- with each $X \in \operatorname{ob}\mathcal{C}$, an object $FX \in \operatorname{ob}\mathcal{D}$;
- with each $f \in \mathcal{C}(X, Y)$, a morphism $Ff \in \mathcal{D}(FX, FY)$,

such that

- $F1_X = 1_{FX}$;
- $F(gf) = Fg \circ Ff$.

LECTURE 4 · 18/10/02

DEFINITION 1.4.2

We define the category **Cat** of small categories:-

- For any category $\mathcal{C}$ there is an identity functor

$$\begin{aligned} 1_{\mathcal{C}}\colon \mathcal{C} &\to \mathcal{C} \\ X &\mapsto X \\ f &\mapsto f \end{aligned}$$

- Composition of functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ with GF defined in the obvious way.

  Similarly we have **CAT**, the category of large categories and functors.

EXAMPLES 1.4.3

1 **Cat** has an initial object 0.
2 **Cat** has a terminal object 1.
3 **Cat** has products; given $\mathcal{C}, \mathcal{D} \in \operatorname{ob} \mathbf{Cat}$, we have the product $\mathcal{C} \times \mathcal{D}$ with

   - objects (c, d), $c \in \mathcal{C}$, $d \in \mathcal{D}$;
   - morphisms (f, g), $f \colon c \longrightarrow c' \in \mathcal{C}$, $g \colon d \longrightarrow d' \in \mathcal{D}$.

DEFINITION 1.4.4

A functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ is *faithful/full/full and faithful* if $\mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$ is injective/surjective/an isomorphism.

EXAMPLES 1.4.5

1 Functors between collections of mathematical objects:

   a forgetful functors:

$$\begin{aligned} \mathbf{Gp} &\to \mathbf{Set} \\ \mathbf{Ring} &\to \mathbf{Set} \\ \mathbf{Ring} &\to \mathbf{Ab} \\ \mathbf{Haus} &\to \mathbf{Top}; \end{aligned}$$

   b free functors:

$$\begin{aligned} \mathbf{Set} &\to \mathbf{Gp} \\ \mathbf{Set} &\to \mathbf{Mnd}; \end{aligned}$$

   c inclusion of subcategories:

$$\begin{aligned} \mathbf{Ab} &\to \mathbf{Gp} \\ \mathbf{Haus} &\to \mathbf{Top}. \end{aligned}$$

2 Functors between mathematical structures:

   a posets $f \colon (P, \leqslant) \longrightarrow (Q, \preccurlyeq)$ is an order-preserving map;
   b groups $f \colon G \longrightarrow H$ is a group homomorphism.

3 Presheaves – a functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is called a *presheaf* on $\mathcal{C}$.
4 Diagrams – a functor $\mathcal{C} \longrightarrow \mathbf{Set}$ is called a *diagram* on $\mathcal{C}$.

Note that a functor will preserve any property that is expressible as a commutative diagram. For example, isomorphisms are preserved by all functors; if f is an isomorphism, then Ff is also.

PROPOSITION

If F is full and faithful, then Ff isomorphic $\Leftrightarrow$ f isomorphic.

PROOF

Let $f \in \mathcal{C}(X, Y)$ such that Ff is an isomorphism. Then $\exists$ inverse $g' \in \mathcal{D}(FY, FX)$ for Ff. Since F is full, then $\exists g \in \mathcal{C}(Y, X)$ such that $g' = Fg$. But now

$$F(fg) = (Ff)(Fg) = 1_{FY}.$$

And $F(1_Y) = 1_{FY}$, so since F is faithful, we have $fg = 1_Y$. Similarly $gf = 1_X$. So g is an inverse for $f \in \mathcal{C}(X, Y)$, i.e. f is an isomorphism. $\square$

1.5 · *Contravariant functors*

DEFINITION 1.5.1

A *contravariant* functor $\mathcal{C} \longrightarrow \mathcal{D}$ is a functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$. That is:

- on objects, $X \longmapsto FX$;
- on morphisms, $X \xrightarrow{f} Y \longmapsto FY \xrightarrow{Ff} FX$;
- identities are preserved;
- $F(g \circ f) = Ff \circ Fg$.

A non-contravariant functor is sometimes referred to as a *covariant* functor.

1.6 · *The Hom functor*

1.6.1 · REPRESENTABLES

Let $\mathcal{C}$ be a locally small category. We have a contravariant functor H_U or $\mathcal{C}(\_, U)$:

$$\begin{aligned} H_U \colon \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ X &\longmapsto \mathcal{C}(X, U) \end{aligned}$$

$$\begin{array}{ccccc} X & & \mathcal{C}(X, U) & & g \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \mathcal{C}(1,g)} & & \big\downarrow \\ Y & & \mathcal{C}(Y, U) & & gf \end{array}$$

Dually, we have a covariant functor H^U or $\mathcal{C}(U, \_)$:

$$\begin{aligned} H^U \colon \mathcal{C} &\longrightarrow \mathbf{Set} \\ X &\longmapsto \mathcal{C}(U, X) \end{aligned}$$

$$\begin{array}{ccccc} X & & \mathcal{C}(U, X) & & g \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \mathcal{C}(f,1)} & & \big\downarrow \\ Y & & \mathcal{C}(U, Y) & & fg \end{array}$$

These are known as *representables*.

1.6.2 · THE Hom FUNCTOR

Again, take $\mathcal{C}$ locally small. Then we have a functor

$$\begin{aligned} H \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\longrightarrow \mathbf{Set} \\ (X, Y) &\longmapsto \mathcal{C}(X, Y) \end{aligned}$$

$$\begin{array}{ccccc} (X, Y) & & \mathcal{C}(X, Y) & & h \\ {\scriptstyle (f,g)}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle \mathcal{C}(f,g)} & & \big\downarrow \\ (X', Y') & & \mathcal{C}(X', Y') & & ghf \end{array}$$

where $f \colon X \longrightarrow X' \in \mathcal{C}^{\mathrm{op}}$ and $g \colon Y \longrightarrow Y' \in \mathcal{C}$.

1.7 · *Natural transformations*

DEFINITION 1.7.1

Let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A *natural transformation* $\alpha\colon F \longrightarrow G$ is a collection of morphisms (known as *components*)

$$\{\, \alpha_X\colon FX \longrightarrow GX \mid X \in \mathcal{C}\,\},$$

such that, $\forall f\colon X \longrightarrow Y \in \mathcal{C}$,

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

commutes (the *naturality condition*).

LECTURE 5 · 21/10/02

DEFINITION 1.7.2

Given categories $\mathcal{C}$ and $\mathcal{D}$, we define the (larger) category $[\mathcal{C}, \mathcal{D}]$ where:

- objects are functors $F\colon \mathcal{C} \longrightarrow \mathcal{D}$;
- morphisms are natural transformations $\alpha\colon F \longrightarrow G$,

such that:

- identities are natural transformations $1_F\colon F \longrightarrow F$ (for any $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ with components $FX \xrightarrow{1_{FX}} FX$;
- for composition, given $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$, then $\beta \circ \alpha$ is the natural transformation with components

$$(\beta \circ \alpha)_X\colon FX \xrightarrow{\beta_X \circ \alpha_X} HX.$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F", ""{name=U, below}] \arrow[r, "G" description, ""{name=M}] \arrow[r, bend right=60, "H"', ""{name=D, above}] & \mathcal{D} \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] \end{tikzcd}$$

So, for example, $[\mathcal{C}, \mathcal{D}](F, G)$ is a collection of natural transformations $F \longrightarrow G$.

DEFINITION 1.7.3

A *natural isomorphism* $\alpha\colon F \longrightarrow G$ is an isomorphism in the functor category; i.e. there exists $\beta\colon G \longrightarrow F$ such that $\alpha \circ \beta = 1_G$ and $\beta \circ \alpha = 1_F$. Note that two natural transformations are equal iff all their components are.

PROPOSITION 1.7.4

$\alpha\colon F \longrightarrow G$ is a natural isomorphism iff each component $\alpha_X\colon FX \longrightarrow GX$ is an isomorphism in $\mathcal{D}$.

PROOF

Suppose α is a natural isomorphism, and let β be its inverse. Then

$$\alpha \circ \beta = 1_G \quad \Rightarrow \quad (\alpha \circ \beta)_X = 1_{GX} \quad \Rightarrow \quad \alpha_X \circ \beta_X = 1_{GX}$$

and

$$\beta \circ \alpha = 1_F \quad \Rightarrow \quad (\beta \circ \alpha)_X = 1_{FX} \quad \Rightarrow \quad \beta_X \circ \alpha_X = 1_{FX}.$$

So β_X is an inverse for α_X for each $X \in \mathcal{C}$. Thus each component is an isomorphism in $\mathcal{D}$.

Conversely, if each component α_X is an isomorphism, then let β_X be the corresponding inverses for each $X \in \mathcal{C}$. Now, given $f \in \mathcal{C}(X, Y)$, we have that

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

commutes; i.e. $(Gf) \circ \alpha_X = \alpha_Y \circ (Ff)$. But now:–

$$\begin{aligned} \beta_Y \circ (Gf) \circ \alpha_X \circ \beta_X &= \beta_Y \circ \alpha_Y \circ (Ff) \circ \beta_X \\ \text{so} \quad \beta_Y \circ (Gf) \circ 1_{GX} &= 1_{FY} \circ (Ff) \circ \beta_X \\ \text{so} \quad \beta_Y \circ (Gf) &= (Ff) \circ \beta_X; \end{aligned}$$

hence

$$\begin{tikzcd} GX \arrow[r, "\beta_X"] \arrow[d, "Gf"'] & FX \arrow[d, "Ff"] \\ GY \arrow[r, "\beta_Y"'] & FY \end{tikzcd}$$

commutes; so we can legitimately define the natural transformation β with components β_X. And clearly β is an inverse for α, so α is a natural isomorphism. □

We can prove similar results that tell us that α is epic/monic iff all its components are.

1.8 · *The 2-category **Cat***

DEFINITION 1.8.1

We define *"horizontal composition"* of natural transformations. We have seen "vertical composition" already:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathcal{D}. \end{tikzcd}$$

But we can also compose:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left=60, "H"] \arrow[r, bend right=60, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "HF"] \arrow[r, bend right=60, "KG"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \mathcal{E}. \end{tikzcd}$$

We define $(\beta * \alpha)_X \colon HFX \longrightarrow KGX$ by

$$HFX \xrightarrow{H\alpha_X} HGX \xrightarrow{\beta_{GX}} KGX$$

or

$$HFX \xrightarrow{\beta_{FX}} KFX \xrightarrow{K\alpha_X} KGX.$$

By the naturality of β, these definitions are equivalent:

$$\begin{tikzcd}
HFX \arrow[r, "\beta_{FX}"] \arrow[d, "H\alpha_X"'] & KFX \arrow[d, "K\alpha_X"] \\
HGX \arrow[r, "\beta_{GX}"'] & KGX
\end{tikzcd}$$

so we can define

$$(\beta * \alpha)_X = \beta_{GX} \circ H\alpha_X = K\alpha_X \circ \beta_{FX}.$$

We consider the following particular case:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left=60, "H"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\Downarrow 1_H"] & \mathcal{E}
\end{tikzcd} \qquad 1_H * \alpha \colon HF \to HG$$

which we will (for convenience) write as:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "H"] & \mathcal{E}
\end{tikzcd} \qquad H\alpha \colon HF \to HG.$$

Similarly we have:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=60, "H"] \arrow[r, bend right=60, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd} \qquad \beta F \colon HF \to KF.$$

PROPOSITION 1.8.2 (THE MIDDLE-4 INTERCHANGE LAW)

Given

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F"] \arrow[r, "G" description] \arrow[r, bend right=70, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha^{(1)}"] \arrow[r, phantom, shift right=3ex, "\Downarrow\alpha^{(2)}"] & \mathcal{D} \arrow[r, bend left=70, "J"] \arrow[r, "K" description] \arrow[r, bend right=70, "L"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\beta^{(1)}"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta^{(2)}"] & \mathcal{E},
\end{tikzcd}$$

we have $(\beta^{(2)} \circ \beta^{(1)}) * (\alpha^{(2)} \circ \alpha^{(1)}) = (\beta^{(2)} * \alpha^{(2)}) \circ (\beta^{(1)} * \alpha^{(1)})$.

PROOF

Consider components. We have

$$\begin{aligned}[(\beta^{(2)} \circ \beta^{(1)}) * (\alpha^{(2)} \circ \alpha^{(1)})]_X &= (\beta^{(2)} \circ \beta^{(1)})_{HX} \circ J(\alpha^{(2)} \circ \alpha^{(1)})_X \\ &= \beta^{(2)}_{HX} \circ \beta^{(1)}_{HX} \circ J\alpha^{(2)}_X \circ J\alpha^{(1)}_X\end{aligned}$$

and

$$[(\beta^{(2)} * \alpha^{(2)}) \circ (\beta^{(1)} * \alpha^{(1)})]_X = \beta^{(2)}_{HX} \circ K\alpha^{(2)}_X \circ \beta^{(1)}_{GX} \circ J\alpha^{(1)}_X.$$

So it is sufficient to prove that $K\alpha^{(2)}_X \circ \beta^{(1)}_{GX} = \beta^{(1)}_{HX} \circ J\alpha^{(2)}_X$. But we have that

$$\begin{tikzcd} JGX \arrow[r, "\beta^{(1)}_{GX}"] \arrow[d, "J\alpha^{(2)}_X"'] & KGX \arrow[d, "K\alpha^{(2)}_X"'] \\ JHX \arrow[r, "\beta^{(1)}_{HX}"'] & KHX \end{tikzcd}$$

commutes (by the naturality of $\beta^{(1)}$), and so we are done. □

DEFINITION 1.8.3

We can now define the 2-category **Cat**, consisting of:

- objects, morphisms and two-cells;
- composition of morphisms;
- horizontal and vertical composition of 2-cells;
- axioms - unit, associativity and middle-4 interchange; "any two ways of composing are the same".

DEFINITION 1.8.4

Given categories $\mathcal{C}$ and $\mathcal{D}$, an *equivalence* consists of:

- functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$, $\mathcal{D} \xrightarrow{G} \mathcal{C}$;
- natural isomorphisms $GF \overset{\alpha}{\Rightarrow} 1_{\mathcal{C}}$, $FG \overset{\beta}{\Rightarrow} 1_{\mathcal{D}}$.

We call β the *inverse up to isomorphism* or the *pseudo-inverse* of α.

DEFINITION 1.8.5

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* on objects iff $\forall\, Y \in \mathcal{D}$, $\exists\, X \in \mathcal{C}$ such that $FX \cong Y \in \mathcal{D}$.

PROPOSITION 1.8.6

F is an equivalence of categories iff it is essentially surjective and full and faithful.

PROOF

Omitted. □

2 · Representability

2.1 · *The Yoneda Embedding*

Recall that for each $A \in \mathcal{C}$, we have the functor $H_A\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. So we have an assignation $A \longmapsto H_A$. We can extend this to a functor, known as the *Yoneda embedding*:–

$$\begin{aligned} H_\bullet\colon \mathcal{C} &\longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \\ A &\longmapsto H_A \\ (f\colon A \longrightarrow B) &\longmapsto (H_f\colon H_A \longrightarrow H_B), \end{aligned}$$

where H_f is the natural transformation with components

$$\begin{aligned} (H_f)_X\colon H_A X &\longrightarrow H_B X \\ \text{i.e. } \mathcal{C}(X, A) &\longrightarrow \mathcal{C}(X, B) \\ h &\longmapsto f \circ h. \end{aligned}$$

We need to check that this is a well-defined natural transformation, i.e. that

$$\begin{tikzcd} \mathcal{C}(Y, A) \arrow[r, "(H_f)_Y = f\circ\_"] \arrow[d, "H_A g = \_\circ g"'] & \mathcal{C}(Y, B) \arrow[d, "H_B g = \_\circ g"] \\ \mathcal{C}(X, A) \arrow[r, "(H_f)_X = f\circ\_"'] & \mathcal{C}(X, B) \end{tikzcd}$$

commutes. But along the two legs we just have:–

$$\begin{tikzcd} h \arrow[r, mapsto] & f \circ h \arrow[d, mapsto] \\ & (f \circ h) \circ g \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} h \arrow[d, mapsto] & \\ h \circ g \arrow[r, mapsto] & f \circ (h \circ g) \end{tikzcd}$$

so the naturality condition just says that composition is associative.

LECTURE 6 · 23/10/02

2.2 · *Representable Functors*

DEFINITION 2.2.1

A functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is *representable* if it is naturally isomorphic to H_A for some $A \in \mathcal{C}$, and a *representation* for F is an object $A \in \mathcal{C}$ together with a natural isomorphism $\alpha\colon H_A \longrightarrow F$.

Dually, a functor $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ is representable if $F \cong H^A$ for some $A \in \mathcal{C}$, and a representation for F is an object A with a natural isomorphism $\alpha\colon H^A \longrightarrow F$.

NOTE

The naturality square says, that $\forall f\colon V \to W \in \mathcal{C}$,

$$\begin{tikzcd} \mathcal{C}(W,A) \arrow[r, "\alpha_W"] \arrow[d, "H_A f = \_\circ f"'] & FW \arrow[d, "Ff"] \\ \mathcal{C}(V,A) \arrow[r, "\alpha_V"'] & FV \end{tikzcd}$$

commutes.

EXAMPLES 2.2.2

1 The forgetful functor $U\colon \mathbf{Gp} \to \mathbf{Set}$ is representable. Take $A = \mathbb{Z}$, and α to be the natural transformation with components:

$$\begin{aligned} \alpha_G\colon H^{\mathbb{Z}}G &\to UG \\ f &\mapsto f(1). \end{aligned}$$

Then we can check that α is natural, and it is an isomorphism, since any homomorphism $f\colon \mathbb{Z} \to G$ is completely determined by $f(1)$.

2 ob: $\mathbf{Cat} \to \mathbf{Set}$ is representable. For let A be 1, the terminal category; then $\mathrm{ob}(\mathcal{C}) \cong \mathbf{Cat}(1, \mathcal{C})$ is a natural isomorphism.

Now, we can make a few suggestive observations about natural transformations $\alpha\colon H_A \to F$. Consider the naturality square

$$\begin{tikzcd} \mathcal{C}(A,A) \arrow[r, "\alpha_A"] \arrow[d, "\_\circ f"'] & FA \arrow[d, "Ff"] \\ \mathcal{C}(V,A) \arrow[r, "\alpha_V"'] & FV \end{tikzcd}$$

We know this commutes; in particular, for the element $1_A \in \mathcal{C}(A,A)$, we have

$$\alpha_V(1_A \circ f) = Ff(\alpha_A(1_A)),$$

so that α is in fact completely determined by $\alpha_A(1_A) \in FA$. So, we would like to define a natural transformation $\alpha\colon H_A \to F$ by setting $\alpha(1_A) = x \in FA$, and $\alpha_V(f) = (Ff)(x)$. If this is indeed a natural transformation, then we will have set up a bijection between FA and the natural transformations $H_A \to F$. Hence we get …

2.3 · *The Yoneda Lemma*

THEOREM 2.3.1 (YONEDA LEMMA)

Let $\mathcal{C}$ be a locally small category, $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. Then there is an isomorphism

$$FA \cong [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F),$$

which is natural in A and F; i.e.

$$\begin{tikzcd} FB \arrow[r] \arrow[d, "Ff"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_B, F)} \arrow[d, "\_\circ H_f"] \\ FA \arrow[r] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FA \arrow[r] \arrow[d, "\theta_A"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)} \arrow[d, "\theta\circ\_"] \\ GA \arrow[r] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, G)} \end{tikzcd}$$

commute, for all $f\colon A \longrightarrow B$ and for all $\theta\colon F \longrightarrow G$ respectively.

PROOF

1 Given $x \in FA$, we define $\widehat{x} \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)$ by components:

$$\begin{aligned} \widehat{X}_V\colon \mathcal{C}(V, A) &\longrightarrow FV \\ f &\longmapsto Ff(x) \end{aligned}$$

We must check the naturality of $\widehat{x}$; given $g\colon W \longrightarrow V$, we need

$$\begin{tikzcd} \mathcal{C}(V, A) \arrow[r, "\widehat{x}_V"] \arrow[d, "\_\circ g"'] & FV \arrow[d, "Fg"] \\ \mathcal{C}(W, A) \arrow[r, "\widehat{x}_W"'] & FW \end{tikzcd}$$

to commute. On elements, we have

$$\begin{tikzcd} f \arrow[r, mapsto] & Ff(x) \arrow[d, mapsto] \\ & Fg\left(Ff(x)\right) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f \arrow[d, mapsto] & \\ f \circ g \arrow[r, mapsto] & F(f \circ g)(x) \end{tikzcd}$$

But $Fg(Ff(x)) = F(f \circ g)(x)$ by the (contravariant) functoriality of F, so the square commutes as required.

2 Given $\alpha \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)$, we define $\widehat{\alpha} \in FA$ by

$$\widehat{\alpha} = \alpha_A(1_A).$$

3 We check $(\widehat{\widehat{\ }}) = (\ )$. Given $x \in FA$,

$$\begin{aligned} \widehat{\widehat{x}} = \widehat{x}_A(1_A) &= F(1_A)(x) \\ &= 1_{FA}(x) \\ &= x. \end{aligned}$$

Given $\alpha \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)$, $\widehat{\widehat{\alpha}}$ is given by components

$$\begin{aligned} \widehat{\widehat{\alpha}}\colon \mathcal{C}(V, A) &\longrightarrow FV \\ f &\longmapsto Ff(\widehat{\alpha}) = Ff(\alpha_A(1_A)). \end{aligned}$$

So we need only check that $\alpha_V(f) = Ff(\alpha_A(1_A))$. We have the following naturality square for α:

$$\begin{tikzcd} \mathcal{C}(A,A) \arrow[r, "\alpha_A"] \arrow[d, "\_\circ f"'] & FA \arrow[d, "Ff"] \\ \mathcal{C}(V,A) \arrow[r, "\alpha_V"'] & FV \end{tikzcd}$$

so on the element $1_A \in \mathcal{C}(A,A)$, we have $\alpha_V(1_A \circ f) = Ff(\alpha_A(1_A))$, as required.

4 We check naturality in A, i.e. that given any $B \xrightarrow{f} A$,

$$\begin{tikzcd} FA \arrow[r, "\widehat{\ }"] \arrow[d, "Ff"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)} \arrow[d, "\_\circ H_f"] \\ FB \arrow[r, "\widehat{\ }"] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_B, F)} \end{tikzcd}$$

commutes. On elements, we have:

$$\begin{tikzcd} x \arrow[r, mapsto] & \widehat{x} \arrow[d, mapsto] \\ & \widehat{x} \circ H_f \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x \arrow[d, mapsto] & \\ Ff(x) \arrow[r, mapsto] & \widehat{Ff(x)}. \end{tikzcd}$$

Now, the former has components

$$\begin{tikzcd} \mathcal{C}(V,B) \arrow[r, "(H_f)_V"] & \mathcal{C}(V,A) \arrow[r, "\widehat{x}_V"] & FV \\ g \arrow[r, mapsto] & f \circ g \arrow[r, mapsto] & F(f \circ g)(x), \end{tikzcd}$$

and the latter

$$\begin{tikzcd} \mathcal{C}(V,B) \arrow[r, "\widehat{Ff(x)}_V"] & FV \\ g \arrow[r, mapsto] & Fg \circ Ff(x). \end{tikzcd}$$

But $(Fg \circ Ff)(x) = F(f \circ g)(x)$ by the functoriality of F; so the naturality square commutes as required.

5 Finally, we must check the naturality in F; given a natural transformation $\theta\colon F \longrightarrow G$, we show that

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "\theta_A"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, F)} \arrow[d, "\theta \circ \_"] \\ GA \arrow[r] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, G)} \end{tikzcd}$$

commutes. We have

$$
\begin{tikzcd}
x \arrow[r, mapsto] & \widehat{x} \arrow[d, mapsto] \\
& \theta \circ \widehat{x}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
x \arrow[d, mapsto] & \\
\theta_A(x) \arrow[r, mapsto] & \widehat{\theta_A(x)}
\end{tikzcd}
$$

with respective components

$$
\begin{aligned}
\mathcal{C}(V,A) &\longrightarrow GA \\
f &\mapsto \theta_V \circ Ff(x)
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
\mathcal{C}(V,A) &\longrightarrow GA \\
f &\mapsto Gf \circ \theta_A(x)
\end{aligned}
$$

But these two are equal by the naturality of θ; so the naturality square commutes as required. $\square$

Dually, for $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$, we have

$$FA \cong [\mathcal{C}, \mathbf{Set}](H^A, F).$$

LECTURE 7 · 25/10/02

THEOREM 2.3.2

The Yoneda embedding is full & faithful.

PROOF

We need to show that $\mathcal{C}(A,B) \xrightarrow{H_\bullet} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, H_B)$ is an isomorphism. By the Yoneda lemma, with $F = H_B$, we have

$$H_B(A) \cong [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](H_A, H_B).$$

So we just need to check that $H_\bullet$ is the same isomorphism as that given by the Yoneda lemma; i.e. that $\widehat{f} = H_f$ or $\widehat{H_f} = f$. But

$$\widehat{H_f} = (H_f)_A(1_A) = f. \qquad \square$$

Note that this shows that, given $f, g\colon A \longrightarrow B$, then $H_f = H_g \Rightarrow f = g$. Also, given $H_A \xrightarrow{h} H_B$, there exists $f\colon A \longrightarrow B$ such that $H_f = h$.

PROPOSITION 2.3.3

$A \cong B \in \mathcal{C}$ implies $\mathcal{C}(X,A) \cong \mathcal{C}(X,B)$ and $\mathcal{C}(A,X) \cong \mathcal{C}(B,X)$, each isomorphism being natural in X.

PROOF

$H_\bullet$ is full and faithful, so $A \cong B \Leftrightarrow H_A \cong H_B$, so $\mathcal{C}(X,A) \cong \mathcal{C}(X,B)$ naturally in X. Similarly for the dual statement. $\square$

2.4 · *Parametrised representability*

Consider $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbf{Set}$. For all $A \in \mathcal{A}$, we get

$$
\begin{aligned}
F(\_, A)\colon \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\
X &\mapsto F(X, A).
\end{aligned}
$$

Suppose each $F(\_, A)$ has a given representation, i.e.

- an object U_A;

- a natural isomorphism $\alpha_A\colon \mathcal{C}(\_, U_A) \to F(\_, A)$.

So we have an assignation $A \mapsto U_A$. Can we extend it to a functor? And are the α_A the components of a natural transformation?

PROPOSITION 2.4.1

Given a functor $F\colon \mathcal{C}^{\mathrm{op}} \times A \to \mathbf{Set}$ such that each $F(\_, A)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ has a representation

$$\alpha_A\colon \mathcal{C}(\_, U_A) \to F(\_, A),$$

then there is a unique way to extend $A \mapsto U_A$ to a functor $U\colon \mathcal{A} \to \mathcal{C}$ such that the α_A are components of a natural transformation $H_\bullet \circ U \to F$.

PROOF

First we construct U on morphisms; i.e. given $f\colon A \to B$, we seek $Uf\colon U_A \to U_B$. In order to satisfy the naturality condition on α, we need

$$\begin{tikzcd}
\mathcal{C}(_, U_A) \arrow[r, "\alpha_A"] \arrow[d] & F(_, A) \arrow[d, "F(_, f)"] \\
\mathcal{C}(_, U_B) \arrow[r, "\alpha_B"'] & F(_, B)
\end{tikzcd}$$

to commute.

Since the horizontal morphisms are isomorphisms, we get a unique morphism on the left $H_{U_A} \to H_{U_B}$ making the diagram commute. Now, the Yoneda embedding is full and faithful, so there exists a unique morphism $U_A \to U_B$ inducing it. Call this Uf. It only remains to check that U is functorial; it will make α a natural transformation by construction.

1 Check $U(1_A) = 1_{UA}$. Note that $U(1_A)$ is the unique morphism making the naturality square commute, so it suffices to check that 1_{UA} makes the square commute.
We have

$$\begin{tikzcd}
\mathcal{C}(_, U_A) \arrow[r, "\alpha_A"] \arrow[d, "1_{U_A} \circ _"'] & F(_, A) \arrow[d, "F(_, 1_A)"] \\
\mathcal{C}(_, U_A) \arrow[r, "\alpha_A"'] & F(_, A)
\end{tikzcd}$$

which commutes as required.

2 We check $U(g \circ f) = Ug \circ Uf$ given $A \xrightarrow{f} B \xrightarrow{g} C$. Consider

$$\begin{tikzcd}
\mathcal{C}(_, U_A) \arrow[r, "\alpha_A"] \arrow[d, "H_{Uf}"'] & F(_, A) \arrow[d, "F(_, f)"] \\
\mathcal{C}(_, U_B) \arrow[r, "\alpha_B"] \arrow[d, "H_{Ug}"'] & F(_, B) \arrow[d, "F(_, g)"] \\
\mathcal{C}(_, U_C) \arrow[r, "\alpha_C"'] & F(_, C)
\end{tikzcd}$$

Each square commutes, so the outside commutes. Now, the composite on the RHS is $F(\_, g \circ f)$, and by definition it induces a unique map $H_{U(g \circ f)}$ on the left such that the diagram commutes. So we must have

$$\begin{aligned} H_{U(g \circ f)} &= H_{Ug} \circ H_{Uf} \\ &= H_{Ug \circ Uf}, \end{aligned}$$

by functorality. But the Yoneda embedding is full and faithful, so we have $U(g \circ f) = Ug \circ Uf$ as required. □

DEFINITION 2.4.2

A *Cartesian closed category* is a category $\mathcal{C}$ equipped with:

- a terminal object T;
- binary objects;
- function spaces.

In fact, in the light of the above results on representability, we can also characterise a Cartesian closed category as containing:

- a representation for the functor $F\colon X \mapsto 1$, since $1 \cong \mathcal{C}(X, T)$ for T a terminal object;
- representations for the functors $F_{A,B}\colon X \longrightarrow \mathcal{C}(X, A) \times \mathcal{C}(X, B)$, since $\mathcal{C}(X, A) \times \mathcal{C}(X, B) \cong \mathcal{C}(X, A \times B)$ naturally in X;
- representations for the functors $F_{B,C}\colon X \longrightarrow \mathcal{C}(X \times B, C)$, since $\mathcal{C}(X \times B, C) \cong \mathcal{C}(X, C^B)$ naturally in X.

We can do even better; using the parametrised representability result, we can:

- from the functor $F\colon (X, (A, B)) \mapsto \mathcal{C}(X, A) \times \mathcal{C}(X, B)$, construct the functor $U\colon (A, B) \mapsto A \times B$;
- from the functor $F\colon (X, (B, C)) \mapsto \mathcal{C}(X \times B, C)$ construct the functor $U\colon (B, C) \mapsto C^B$.

LECTURE 8 · 25/10/02

3 · Limits & colimits

3.1 · *Introduction*

Consider any drawable diagram contained within some category $\mathcal{D}$; for example

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Then a *limit* over this diagram is a universal cone:

3.1.1 · CONES

A *cone over a diagram* consists of:

- a vertex - an object in $\mathcal{D}$;
- projections - a morphism from the vertex to each object of the diagram,

such that all the resulting triangles commute:

$$\begin{tikzcd} & & & \bullet \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\ \bullet \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet \arrow[rr] & & \bullet \arrow[rr] & & \bullet \end{tikzcd}$$

3.1.2 · LIMITS AS UNIVERSAL CONES

Informally, something is universal with respect to a property if any other thing with that property factors through it uniquely. A limit is a universal cone over a diagram; that is, a cone such that any other cone factors through it uniquely. For example:

$$\begin{tikzcd}[column sep=small] & & & L \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\ \bullet \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet \arrow[rr] & & \bullet \arrow[rr] & & \bullet \end{tikzcd} \quad \text{s.t. given } Y \text{ with} \quad \begin{tikzcd}[column sep=small] & Y \arrow[rrrr, dotted, "\varphi"] \arrow[dl] \arrow[dr] \arrow[drrr] \arrow[drrrrr] & & & & L \arrow[dlllll] \arrow[dlll] \arrow[dl] \arrow[dr] & \\ \bullet \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet \arrow[rr] & & \bullet \arrow[rr] & & \bullet \end{tikzcd}$$

there exists unique φ such that all the triangles commute. As before, the limit is unique up to unique isomorphism.

3.1.3 · LIMITS OVER d

Let $\mathbb{I}$ be a small category ($\mathbb{I}$ is a generalisation of our "drawable diagram"), and let D be a functor $\mathbb{I} \longrightarrow \mathcal{D}$. Then we have the *cone over D*:

- a vertex $L \in \mathcal{D}$;
- for each object $I \in \mathbb{I}$, a morphism $k_I \colon L \longrightarrow DI$

such that, for all $u \colon I \longrightarrow I' \in \mathbb{I}$,

$$\begin{tikzcd} & L \arrow[dl, "k_I"'] \arrow[dr, "k_{I'}"] & \\ DI \arrow[rr, "Du"'] & & DI' \end{tikzcd}$$

commutes. We write $(L \xrightarrow{k_X} DI)_{I \in \mathbb{I}}$.

A limit is a universal cone, and the universal property says: given a cone $(Y \xrightarrow{p_X} DI)_{I \in \mathbb{I}}$, there exists a unique morphism $f \colon Y \longrightarrow L$ such that "all triangles commute", i.e., for all $I \in \mathbb{I}$,

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p_I"'] & & L \arrow[dl, "k_I"] \\ & DI & \end{tikzcd}$$

commutes.

3.2 · *Some specific limits*

3.2.1 · PRODUCTS

A product is a limit of shape $\mathbb{I}$ with $\mathbb{I}$ discrete. So, for example, we have

$$\begin{tikzcd}[column sep=small] & & & L \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\ DI & & DI' & & DI'' & & DI''' \end{tikzcd}$$

our cone, where $DI, \cdots \in \operatorname{ob} \mathcal{D}$. The universal property says, given any other cone from L', say, then

$$\begin{tikzcd}[column sep=small] & L' \arrow[rrrr, dotted] \arrow[dl] \arrow[dr] \arrow[drrr] \arrow[drrrrr] & & & & L \arrow[dlllll] \arrow[dlll] \arrow[dl] \arrow[dr] & \\ DI & & DI' & & DI'' & & DI''' \end{tikzcd}$$

has a unique morphism $L' \longrightarrow L$ such that every triangle commutes. We write

$$\prod_{I \in \mathbb{I}} DI \xrightarrow{p_I} DI.$$

We have already seen the product over the empty set, i.e. a terminal object, and the product over $\{\bullet, \bullet\}$; that is, a binary product.

3.2.2 · EQUALISERS

An equaliser is a limit of shape $\bullet \rightrightarrows \bullet$. A diagram of this shape in $\mathcal{D}$ is of the form

$$A \overset{f}{\underset{g}{\rightrightarrows}} B.$$

A cone over this diagram is

$$\begin{tikzcd} & E \arrow[dl, "e"'] \arrow[dr, "m"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B. \end{tikzcd}$$

Note that $m = fe = ge$ as all triangles commute; so in fact we can rewrite this more simply as

$$E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B \qquad \text{such that } fe = ge.$$

An equaliser is the universal such; so given any $C \xrightarrow{h} A \overset{f}{\underset{g}{\rightrightarrows}} B$ such that $fh = gh$, then there exists a unique factorisation:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ C \arrow[u, "\exists!\bar{h}"] \arrow[ur, "h"'] & & \end{tikzcd}$$

such that $h = e\bar{h}$.

3.2.3 · PULLBACKS

A pullback is a limit of shape

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

A diagram of this shape in $\mathcal{D}$ is

$$\begin{tikzcd} & W \arrow[d, "g"'] \\ U \arrow[r, "f"'] & V. \end{tikzcd}$$

A cone over this diagram is

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "a"'] & W \arrow[d, "f'"] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$

commuting (really, there is a projection $c\colon Z \to V$, but we must have $c = fa = gb$). A pullback is the universal such; so given any commutative square

$$\begin{tikzcd} Z \arrow[r, "b"] \arrow[d, "a"'] & W \arrow[d, "g"] \\ U \arrow[r, "f"'] & V, \end{tikzcd}$$

we have

$$\begin{tikzcd} Z \arrow[dr, dotted, "\exists! h"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\ & P \arrow[r, "f'"] \arrow[d, "g'"'] & W \arrow[d, "g"] \\ & U \arrow[r, "f"'] & V \end{tikzcd}$$

a unique h such that $g'h = a$, and $f'h = b$. We say that g' is a pullback for g over f, and that f' is a pullback for f over g.

LECTURE 9 · 30/10/02

3.3 · *Limits — formally*

DEFINITION 3.3.1

Given $Y \in \mathcal{D}$, we define the *constant functor* ΔY:

$$\begin{aligned} \Delta Y\colon \mathbb{I} &\longrightarrow \mathcal{D} \\ I &\longmapsto Y \\ f &\longmapsto 1_Y. \end{aligned}$$

From this we get a functor:

$$\begin{aligned} \Delta\_ : \mathcal{D} &\longrightarrow [\mathbb{I}, \mathcal{D}] \\ Y &\longmapsto \Delta Y \end{aligned}$$

$$\begin{tikzcd} X \arrow[d, "f"'] & \Delta X \arrow[d, "\Delta f"] \\ Y \arrow[ur, phantom, "\longmapsto"] & \Delta Y \end{tikzcd}$$

with every component of Δf being f.

DEFINITION 3.3.2

A *limit* for $D\colon \mathbb{I} \longrightarrow \mathcal{D}$ is a representation for the functor

$$[\mathbb{I}, \mathcal{D}](\Delta\_, D)\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

That is, an object $L \in \mathcal{D}$ and a natural isomorphism α with

$$H_L \overset{\alpha}{\cong} [\mathbb{I}, \mathcal{D}](\Delta\_, D).$$

We write $L = \lim_{\longleftarrow \mathbb{I}} D = \int_I DI$.

So we have an isomorphism

$$\mathcal{D}(\_, \textstyle\int_I DI) \cong [\mathbb{I}, \mathcal{D}](\Delta\_, D).$$

Let us make explicit what the functor on the right hand side does; call it F. Then:

$$\begin{aligned} F\colon \mathcal{D}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ Y &\longmapsto [\mathbb{I}, \mathcal{D}](\Delta Y, D) \end{aligned}$$

$$\begin{tikzcd} Y \arrow[d, "f"'] & \longmapsto & {[\mathbb{I}, \mathcal{D}](\Delta X, D)} \arrow[d, "Ff"] & \theta \arrow[d, mapsto] \\ X & & {[\mathbb{I}, \mathcal{D}](\Delta Y, D)} & \theta \circ \Delta f. \end{tikzcd}$$

Now, what does a natural transform $\Delta Y \xrightarrow{k} D$ look like? We have:

- for each $I \in \mathbb{I}$, a morphism

$$\begin{aligned} k_I\colon (\Delta Y)I &\longrightarrow DI \\ Y &\longrightarrow DI; \end{aligned}$$

- for all $u\colon I \longrightarrow I'$ in $\mathbb{I}$,

$$\begin{tikzcd} (\Delta Y)I \arrow[r] \arrow[d, "(\Delta Y)u"'] & DI \arrow[d, "Du"] \\ (\Delta Y)I' \arrow[r] & DI' \end{tikzcd}$$

commutes by naturality; i.e.

$$\begin{tikzcd} & Y \arrow[dl, "k_I"'] \arrow[dr, "k_{I'}"] & \\ DI \arrow[rr, "Du"'] & & DI' \end{tikzcd}$$

commutes.

So such a natural transformation is precisely a cone over D with Y as the vertex. Now, consider a representation as above, and let α be its natural isomorphism. Then we have

$$\begin{aligned} \alpha_Y\colon \mathcal{D}(Y, L) &\longrightarrow [\mathbb{I}, \mathcal{D}](\Delta Y, D) \\ f &\longmapsto Ff(\alpha_L 1_L); \end{aligned}$$

i.e., the natural transformation is completely determined by $\alpha_L 1_L$.

Now, we have a cone given by $\alpha_L 1_L = (k_I)_{I \in \mathbb{I}}$, say. So given any other Y and $Y \xrightarrow{f} L$ on the left hand side, we have $Ff(\alpha_L 1_L)$ with components $k_I \circ f$; hence we have a bijective correspondence

$$\begin{array}{c} \text{morphisms} \\ Y \xrightarrow{f} L \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \text{cones over } D \\ (k_I \circ f)_{I \in \mathbb{I}} \end{array}$$

i.e., starting on the right hand side, given any cone $(p_I)_{I \in \mathbb{I}}$, there exists a unique morphism $f \colon Y \longrightarrow L$ such that $p_I = k_I \circ f$ for all I; thus $(k_I)_{I \in \mathbb{I}}$ is a universal cone over D.

Note that any isomorphism on the left hand side will give rise to a universal cone.

DEFINITION 3.3.3

If a limit exists for all functors from $D \colon \mathbb{I} \longrightarrow \mathcal{D}$, we say $\mathcal{D}$ has all limits of shape $\mathbb{I}$.

If $\mathcal{D}$ has all limits of shape $\mathbb{I}$ for all small/finite categories $\mathbb{I}$, we say $\mathcal{D}$ has all small/finite limits or that $\mathcal{D}$ is (finitely) complete.

3.4 · *Limits in* ***Set***

THEOREM 3.4.1

Set has all small limits.

PROOF

We seek a limit for $F \colon \mathbb{I} \longrightarrow$ **Set**. We define L, a set of tuples $\subseteq \prod_{I \in \mathbb{I}} FI$ by taking all tuples $(x_I)_{I \in \mathbb{I}}$ satisfying:

- $\forall I \in \mathbb{I}$, $x_I \in FI$;
- $\forall I \xrightarrow{u} I'$, $Fu(x_I) = x_{I'}$.

We have projections

$$\begin{aligned} L &\xrightarrow{p_I} FI \\ (x_I)_{I \in \mathbb{I}} &\longmapsto x_I \end{aligned}$$

for each $I \in \mathbb{I}$. We now show that this is a minimal cone:

1 It is a cone; we need to show, for all $u \colon I \longrightarrow I'$, that

$$\begin{tikzcd} & L \arrow[dl, "p_{I'}"'] \arrow[dr, "p_I"] & \\ FI' & & FI \arrow[ll, "Fu"] \end{tikzcd}$$

commutes. On elements we have

$$\begin{tikzcd} & (x_I)_{I \in \mathbb{I}} \arrow[dr, mapsto] \\ Fu(x_I) & & x_I \arrow[ll, mapsto] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & (x_I)_{I \in \mathbb{I}} \arrow[dl, mapsto] \\ x_{I'} & \end{tikzcd}$$

so we are done here, since $Fu(x_I) = x_{I'}$.

2 It is universal: we show that every cone factors through it uniquely. So consider a cone $(Z \xrightarrow{q_I} FI)_{I \in \mathbb{I}}$; so

$$\begin{tikzcd} & Z \arrow[dl, "q_{I'}"'] \arrow[dr, "q_I"] & \\ FI' & & FI \arrow[ll, "Fu"'] \end{tikzcd}$$

commutes; that is, for all $y \in Z$, $Fu(q_I(y)) = q_{I'}(y)$. We seek a unique factorisation making the following diagram commute for all I:

$$\begin{tikzcd} L \arrow[dr, "p_I"'] & & Z \arrow[ll, "h"'] \arrow[dl, "q_I"] \\ & FI & \end{tikzcd}$$

On elements, this would give

$$\begin{tikzcd} h(y) \arrow[dr, mapsto] & & y \arrow[ll, mapsto] \arrow[dl, mapsto] \\ & q_I(y) & \end{tikzcd}$$

So, writing $h(y) = (a_I)_{I \in \mathbb{I}}$, we must have $a_I = q_I(y)$. So define h by $h(y) = (q_I(y))_{I \in \mathbb{I}}$. It remains to check that $h(y) \in L$, so that for all $u\colon I \longrightarrow I'$, $Fu(a_I) = a_{I'}$; i.e.

$$Fu(q_I(y)) = q_{I'}(y),$$

which follows since $(Z \xrightarrow{q_I} FI)_{I \in \mathbb{I}}$ is a cone. □

LECTURE 10 · 01/11/02

3.5 · *Limits in other categories*

THEOREM 3.5.1

If a category $\mathcal{D}$ has all small products and equalisers, then $\mathcal{D}$ has all small limits.

PROOF

Given a diagram $D\colon \mathbb{I} \longrightarrow \mathcal{D}$, $\mathbb{I}$ small, we seek a limit in $\mathcal{D}$. The idea of the proof is to construct it as an equaliser $E \xrightarrow{e} P \underset{g}{\overset{f}{\rightrightarrows}} Q$, where P and Q are certain products over the DI.

1 Put

$$P = \prod_{I \in \mathbb{I}} D_I$$

with projections $P \xrightarrow{p_I} DI$; this is a small product, so exists.

2 Put

$$Q = \prod_{u\colon I \longrightarrow J \in \mathbb{I}} DJ$$

with projections $Q \xrightarrow{q_U} DJ$; again, a small product, so exists.

3 Induce f by the universal property of Q as follows: for all $u\colon I \longrightarrow J$, we have $p_J\colon P \longrightarrow DJ$ inducing a unique $f\colon P \longrightarrow Q$ such that $\forall u$,

$$q_U \circ f = p_J. \qquad (1)$$

$$\begin{tikzcd}
P \arrow[rr, dotted, "!f"] \arrow[dr, "p_J"'] & & Q \arrow[dl, "q_U"] \\
& DJ &
\end{tikzcd}$$

4 Induce g by the universal property of product Q (differently) as follows: for all $u\colon I \longrightarrow J$, we have $Du \circ p_I\colon P \longrightarrow DJ$ inducing a unique $g\colon P \longrightarrow Q$ such that, for all u,

$$q_u \circ g = Du \circ p_I. \tag{2}$$

$$\begin{tikzcd}
P \arrow[r, dotted, "!g"] \arrow[d, "p_I"'] & Q \arrow[d, "q_U"] \\
DI \arrow[r, "Du"'] & DJ
\end{tikzcd}$$

5 Take equaliser $E \xrightarrow{e} P \underset{g}{\overset{f}{\rightrightarrows}} Q$; so in particular

$$fe = ge. \tag{3}$$

Claim that $(E \xrightarrow{p_I \circ e} DI)_{I \in \mathbb{I}}$ gives a universal cone over D.

6 First we show it is a cone; i.e. for all $u\colon I \longrightarrow J$,

$$Du \circ p_I \circ e = p_J \circ e \tag{4}$$

This is true, since

$$\begin{aligned}
Du \circ p_i \circ e &= q_u \circ g \circ e && \text{by (2)} \\
&= q_u \circ f \circ e && \text{by (3)} \\
&= p_J \circ e && \text{by (1)}
\end{aligned}$$

It remains to show that this cone is universal; i.e. given any cone $(V \xrightarrow{v_I} DI)_{I \in \mathbb{I}}$, we seek a unique $x\colon V \longrightarrow E$ such that for all $I \in \mathbb{I}$, $p_I \circ e \circ x = v_I$.

$$\begin{tikzcd}
& V \arrow[dl, dotted, "x"'] \arrow[dd, "v_I"] \\
E \arrow[dr, "p_I \circ e"'] & \\
& DI
\end{tikzcd}$$

We will construct a diagram

$$\begin{tikzcd}
V \arrow[d, dotted, "x"'] \arrow[dr, dotted, "k"] \arrow[drr, dotted, bend left, "m"] \arrow[dd, bend right, "v_I"'] & & \\
E \arrow[r, "e"] & P \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dl, "p_I"] & Q \\
DI & &
\end{tikzcd}$$

So suppose we are given such a cone $(V \xrightarrow{v_I} DI)_{I \in \mathbb{I}}$. So for all $u\colon I \longrightarrow J$,

$$Du \circ v_I = v_J. \tag{5}$$

7 Induce $k\colon V \longrightarrow P$ by the universal property of P: for all $I \in \mathbb{I}$, we have $V \xrightarrow{v_I} DI$ inducing a unique $k\colon V \longrightarrow P$ such that, for all I,

$$p_I \circ k = v_I. \tag{6}$$

8 Induce $x\colon V \longrightarrow E$ by the universal property of the equaliser; in order to do this, we must first show that $fk = gk$. Now, for all $u\colon I \longrightarrow J$, we have $V \underset{v_J}{\longrightarrow} DJ$ inducing a unique $m\colon V \longrightarrow Q$ such that

$$q_u \circ m = v_J. \tag{7}$$

But fk and gk both satisfy this condition, since, for all u,

$$\begin{aligned} q_u \circ f_k &= p_J \circ k && \text{by (1)} \\ &= v_J && \text{by (6)} \end{aligned}$$

and

$$\begin{aligned} q_u \circ g_k &= Du \circ p_I \circ k && \text{by (2)} \\ &= D_u \circ v_I && \text{by (6)} \\ &= v_J && \text{by (5)} \end{aligned}$$

Hence $fk = gk$; so we can induce a unique $x\colon V \longrightarrow E$ such that

$$e \circ x = k. \tag{8}$$

9 We now check that x is a factorisation for the cones. So given $I \in \mathbb{I}$,

$$\begin{aligned} p_I \circ e \circ x &= p_I \circ k && \text{by (8)} \\ &= v_I && \text{by (6)} \end{aligned}$$

so we have the desired result.

10 Finally, we show that x is unique with this property; suppose we have a morphism $y\colon V \longrightarrow E$ such that, for all I,

$$p_I \circ e \circ y = v_I. \tag{9}$$

Now by construction x is unique such that $ex = k$, so we seek to show also $ey = k$. By construction, k is unique such that for all I, $p_I \circ k = v_I$ (by (6)); but (9) says that ey also satisfies this. Hence $ey = k$, so $y = x$ and we are done. □

3.6 · *Colimits*

DEFINITION 3.6.1

A *colimit* for a diagram $D\colon \mathbb{I} \longrightarrow \mathcal{D}$ is a representation

$$\mathcal{D}(\textstyle\int^{I} DI, \_\,) \cong [\mathbb{I}, \mathcal{D}](D, \Delta\_\,).$$

So a colimit for $D\colon \mathbb{I} \longrightarrow \mathcal{D}$ is essentially a limit of $D^{\mathrm{op}}\colon \mathbb{I}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$. If D has all small colimits, we say it is *cocomplete*.

LECTURE 11 · 04/11/02

3.7 · *Parametrised limits*

Recall two results:

1 Given a diagram $D\colon \mathbb{I} \longrightarrow \mathcal{D}$, a limit for D is a representation

$$\mathcal{D}(\_\,, \textstyle\int_I DI) \cong [\mathbb{I}, \mathcal{D}](\Delta\_\,, D)$$

2 Given a functor $X\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbf{Set}$ such that each $X(\_\,, A)$ has a representation

$$\alpha_A\colon \mathcal{C}(\_\,, U_A) \cong X(\_\,, A)$$

then there is a unique way to extend $A \mapsto U_A$ to a functor such that

$$\mathcal{C}(Y, U_A) \cong X(Y, A)$$

naturally in Y and A, with components of the implied natural transformation given by α_A.

PROPOSITION 3.7.1

Define $F\colon \mathbb{I} \times \mathcal{A} \longrightarrow \mathcal{D}$ such that each $F(\_\,, A)\colon \mathbb{I} \longrightarrow \mathcal{D}$ has a specified limit in $\mathcal{D}$:

$$\mathcal{D}(\_\,, \textstyle\int_I F(I, A)) \cong [\mathbb{I}, \mathcal{D}](\Delta\_\,, F(\_\,, A)).$$

Then there is a unique way to extend $A \mapsto \int_I F(I, A)$ to a functor $\mathcal{A} \longrightarrow \mathcal{D}$ such that

$$\mathcal{D}(Y, \textstyle\int_I F(I, A)) \cong [\mathbb{I}, \mathcal{D}](\Delta Y, F(\_\,, A))$$

naturally in Y and A.

PROOF

Simple application of parametrised representability. □

APPLICATION 3.7.2

Suppose $\mathcal{D}$ has chosen limits of shape $\mathbb{I}$. Consider the evaluation functor

$$\begin{aligned} \mathcal{E}\colon \mathbb{I} \times [\mathbb{I}, \mathcal{D}] &\longrightarrow \mathcal{D} \\ (I, D) &\longmapsto DI \end{aligned}$$

Then $\mathcal{E}(\_\,, D)$ has a limit for each D, $\int_I DI$. By parametrised limits, we get a functor

$$\begin{aligned} \textstyle\int_I\colon [\mathbb{I}, \mathcal{D}] &\longrightarrow \mathcal{D} \\ D &\longmapsto \textstyle\int_I DI \end{aligned}$$

such that $\mathcal{D}(Y, \int_I DI) \cong [\mathbb{I}, \mathcal{D}](\Delta Y, D)$ naturally in Y and D.

APPLICATION 3.7.3

We can restate the definition of a limit to get

$$\mathcal{D}(Y, \textstyle\int_I DI) \cong \textstyle\int_I \mathcal{D}(Y, DI).$$

What does this mean?

1 The right hand side is the limit of the functor

$$\begin{aligned} \mathcal{D}(Y, D\_)\colon \mathbb{I} &\longrightarrow \mathbf{Set} \\ I &\longmapsto \mathcal{D}(Y, DI) \end{aligned}$$

$$\begin{tikzcd} I \arrow[d, "u"'] & & \mathcal{D}(Y, DI) \arrow[d, "Du\circ\_"] \\ I' & \longmapsto & \mathcal{D}(Y, DI') \end{tikzcd}$$

Set is complete, so this certainly has a limit. What does $\int_I \mathcal{D}(Y, DI)$ look like? Well, it is all tuples $(\alpha_I)_{I\in\mathbb{I}}$ such that

$$\forall I, \alpha_I \in \mathcal{D}(Y, DI)$$

and

$$\forall u\colon I \longrightarrow I', Du \circ \alpha_I = \alpha_{I'}.$$

So this is precisely a cone over D; i.e.

$$\textstyle\int_I \mathcal{D}(Y, DI) = [\mathbb{I}, \mathcal{D}](\Delta Y, D)$$

2 Observe that by parametrised limits, we have a functor

$$Y \longmapsto \textstyle\int_I \mathcal{D}(Y, DI)$$

So

$$\textstyle\int_I \mathcal{D}(Y, DI) = [\mathbb{I}, \mathcal{D}](\Delta Y, D) \cong \mathcal{D}(Y, \int DI)$$

naturally in Y and D.

3.8 · *Preservation, reflection and creation of limits*

Let $\mathbb{I} \xrightarrow{D} \mathcal{D} \xrightarrow{F} \mathcal{E}$. We can consider limits over D and limits over FD.

DEFINITION 3.8.1

Suppose we have a limit cone for D

$$(\textstyle\int_I DI \xrightarrow{k_I} DI)_{I\in\mathbb{I}}$$

We say F *preserves* this limit if

$$(F \textstyle\int_I DI \xrightarrow{Fk_I} FDI)_{I\in\mathbb{I}}$$

is a limit cone for FD in $\mathcal{E}$. Note that it must preserve projections.

DEFINITION 3.8.2

Suppose $FD\colon \mathbb{I} \longrightarrow \mathcal{E}$ has a limit cone. We say F *reflects* this limit if any cone that goes to a limit cone was already a limit cone itself. That is, given a cone

$$(Z \xrightarrow{f_I} DI)_{I\in\mathbb{I}}$$

such that $(FZ \xrightarrow{Ff_I} FDI)_{I\in\mathbb{I}}$ is a limit cone for FD, then $(Z \xrightarrow{f_I} DI)_{I\in\mathbb{I}}$ is also a limit cone.

DEFINITION 3.8.3

Suppose $FD\colon \mathbb{I} \longrightarrow \mathcal{E}$ has a limit cone. We say F *creates* this limit if there exists a cone $(Z \xrightarrow{f_I} DI)_{I\in\mathbb{I}}$ such that $(FZ \xrightarrow{Ff_I} FDI)_{I\in\mathbb{I}}$ is a limit cone for FD, and additionally F reflects limits. That is, given a limit for FD, there is a unique-up-to-isomorphism lift to a limit for D.

LECTURE 12 · 06/11/02

3.9 · *Examples of preservation, reflection and creation*

PROPOSITION 3.9.1

Representable functors preserve limits.

PROOF

We consider

$$\mathbb{I} \xrightarrow{D} \mathcal{C} \xrightarrow{H^U} \mathbf{Set}$$
$$I \longmapsto DI \longmapsto \mathcal{C}(U, DI)$$

Given a limit cone for D,

$$(\textstyle\int_I DI \xrightarrow{k_I} DI)_{I\in\mathbb{I}},$$

we need to show that

$$\mathcal{C}(U, \textstyle\int_I DI) \xrightarrow{k_I\circ} \mathcal{C}(U, DI)$$

is a limit cone for $\mathcal{C}(U, D\_)$. Certainly, $\mathcal{C}(U, \int_I DI) \cong \int_I \mathcal{C}(U, DI)$. And for projections

$$\mathcal{C}(U, \textstyle\int_I DI) \cong [\mathbb{I}, \mathcal{C}](\Delta U, D) = \int_I \mathcal{C}(U, DI)$$
$$f \longmapsto k_I \circ f$$

so we are done. Dually, we have

$$\mathcal{C}(\textstyle\int^I DI, U) \cong \int_I \mathcal{C}(DI, U)$$

so H_U takes a colimit in $\mathcal{C}$ to a limit in **Set**; and hence takes a limit in $\mathcal{C}^{\mathrm{op}}$ to a limit in **Set**. Thus H_U also preserves limits. □

PROPOSITION 3.9.2

A full and faithful functor preserves limits.

PROOF

Consider $\mathbb{I} \xrightarrow{D} \mathcal{C} \xrightarrow{F} \mathcal{E}$, with F full and faithful, and let $(Z \xrightarrow{f_I} DI)_{I\in\mathbb{I}}$ be a cone such that F of it is a limit cone for FD. We need to show that this cone itself is a limit.

Now, given any other cone $(W \xrightarrow{g_I} DI)_{I\in\mathbb{I}}$, we seek a unique h such that $g_I = f_I \circ h$ for all $I \in \mathbb{I}$. So

1. Since $F(Z \xrightarrow{f_I} DI)$ is a limit, there exists unique m such that $Fg_I = Ff_i \circ m$ for all $I \in \mathbb{I}$.
2. Since F is full, there exists $h\colon W \longrightarrow Z$ such that $Fh = m$.
3. Check commuting condition: we know that, for all $I \in \mathbb{I}$, $Fg_I = Ff_i \circ Fh$, i.e. $Fg_I = F(f_i \circ h)$. Hence $f_I \circ h = g_I$ since F is faithful.
4. Suppose there is a k such that for all $I \in \mathbb{I}$, $f_I \circ k = g_I$. Then $Ff_I \circ Fk = Fg_I$ for all I; but we have that m is the unique morphism such that $Ff_i \circ m = Fg_I$; hence $Fk = m = Fh$, so $k = h$ (as F faithful), and we are done. □

4 · Ends and coends

4.1 · Dinaturality

DEFINITION 4.1.1

Given functors $F, G\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$, a *dinatural transform* $\alpha\colon F \longrightarrow G$ consists of, for each $U \in \mathcal{C}$, a component

$$\alpha_U\colon F(U,U) \longrightarrow G(U,U)$$

such that for all $f\colon U \longrightarrow V$,

$$\begin{tikzcd}
& F(U,U) \arrow[r, "\alpha_U"] & G(U,U) \arrow[dr, "{G(1,f)}"] & \\
F(U,V) \arrow[ur, "{F(f,1)}"] \arrow[dr, "{F(1,f)}"'] & & & G(U,V) \\
& F(V,V) \arrow[r, "\alpha_V"'] & G(V,V) \arrow[ur, "{G(f,1)}"'] &
\end{tikzcd}$$

commutes.

Note that there is no sensible composition of dinatural transformation, and hence $\mathrm{Dinat}(F,G)$ is just a set.

4.2 · Ends and coends

Recall that a limit for $D\colon \mathbb{I} \longrightarrow \mathcal{D}$ is a representation for $[\mathbb{I}, \mathcal{D}](\Delta\_, D) = \mathrm{Nat}(\Delta\_, D)$, such that

$$\mathcal{D}(Y, \textstyle\int_I DI) \cong \mathrm{Nat}(\Delta Y, D) \qquad \text{naturally in } Y.$$

DEFINITION 4.2.1

An *end* for $F\colon \mathbb{I}^{\mathrm{op}} \times \mathbb{I} \longrightarrow \mathcal{D}$ is a representation for the functor

$$\mathrm{Dinat}(\Delta\_, F)\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

so that

$$\mathcal{D}(Y, \textstyle\int_I F(I,I)) \cong \mathrm{Dinat}(\Delta Y, F) \qquad \text{naturally in } Y.$$

Dually, a *coend* for F is just a representation for $\mathrm{Dinat}(F, \Delta\_)\colon \mathcal{D} \longrightarrow \mathbf{Set}$ so

$$\mathcal{D}(\textstyle\int^I F(I,I), Y) \cong \mathrm{Nat}(F, \Delta Y) \qquad \text{naturally in } Y.$$

REMARK

Ends are in fact just a special sort of limit; any end can be expressed as a limit.

4.3 · Ends in **Set**

Recall a limit in **Set** for $D\colon \mathbb{I} \longrightarrow \mathbf{Set}$ is given by

$$\{\, (x_I)_{I \in \mathbb{I}} \mid \forall I, x_i \in DI, \forall u\colon I \longrightarrow I', Du(x_I) = x_{I'} \,\}.$$

An end in **Set** for $X\colon \mathbb{I}^{\mathrm{op}} \times \mathbb{I} \longrightarrow \mathbf{Set}$ is given by

$$\{\, (x_I)_{I \in \mathbb{I}} \mid \forall I, x_i \in X(I,I), \forall f\colon I \longrightarrow I', X(1,f)(x_I) = X(f,1)(x_{I'}) \,\}.$$

4.4 · *Key observations*

OBSERVATION 4.4.1

Parametric results follow, so we can use ends in **Set** to restate the definition of (co)ends. Consider

$$\begin{aligned} X_V \colon \mathbb{I}^{\mathrm{op}} \times \mathbb{I} &\longrightarrow \mathbf{Set} \\ (I, J) &\longmapsto \mathcal{D}(V, F(I, J)) \end{aligned}$$

We have an end in **Set**

$$\textstyle\int_I X_V(I, I) \cong \int_I \mathcal{D}(V, F(I, I)) = \mathrm{Dinat}(\Delta V, F)$$

So we get:

$$\begin{aligned} \text{End: } \mathcal{D}(V, \textstyle\int_I F(I, I)) &\cong \textstyle\int_I \mathcal{D}(V, F(I, I)) \\ \text{Coend: } \mathcal{D}(\textstyle\int^I F(I, I), V) &\cong \textstyle\int_I \mathcal{D}(F(I, I), V) \end{aligned}$$

OBSERVATION 4.4.2

The set $[\mathbb{C}, \mathcal{D}](F, G)$ is an end in **Set**. For consider

$$\begin{aligned} X \colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} &\longrightarrow \mathbf{Set} \\ (U, V) &\longmapsto \mathcal{D}(FU, GV) \end{aligned}$$

Then $\int_U X(U, U) = \int_U \mathcal{D}(FU, GI)$ is just

$$\{\, (\alpha_U)_{U \in \mathbb{C}} \mid \alpha_U \colon FU \longrightarrow GU \text{ and } \forall f \colon U \longrightarrow U', X(1, f)(\alpha_U) = X(f, 1)(\alpha_{U'}) \,\}.$$

But now

$$Gf \circ \alpha_U = X(1, f)(\alpha_U) = X(f, 1)(\alpha_{U'}) = \alpha_{U'} \circ Ff$$

so this is just a naturality condition on the α_U's; and hence we have

$$\textstyle\int_U X(U, U) = \int_U \mathcal{D}(FU, GI) = [\mathbb{C}, \mathcal{D}](F, G).$$

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OBSERVATION 4.4.3

We can restate the Yoneda lemma. Recall that if $X \colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, we have

$$\begin{aligned} X(U) &\cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](H_U, X) \\ &\cong \textstyle\int_V [H_U(V), X(V)] && \text{where } [\ ,\ ] \text{ means morphisms in } \mathbf{Set} \\ &\cong \textstyle\int_V [\mathbb{C}(V, U), X(V)] \end{aligned}$$

4.5 · *Applications*

Consider a functor $F \colon \mathbb{I} \longrightarrow [\mathbb{C}, \mathcal{D}]$. What does a limit cone for this look like? We have

$$(L \xrightarrow{\alpha_I} FI)_{I \in \mathbb{I}}$$

with L a functor and α_I a natural transformation $L \longrightarrow FI$ with components $(\alpha_I)_C \colon LC \longrightarrow FI(C)$.

Now, given $C \in \mathbb{C}$, we can evaluate the whole cone at C:

$$(LC \xrightarrow{\alpha_{IC}} FI(C))_{I \in \mathbb{I}}$$

Now if this is a limit cone in $\mathcal{D}$ for

$$\begin{aligned} F_C\colon \mathbb{I} &\longrightarrow \mathcal{D} \\ I &\longmapsto FI(C) \end{aligned}$$

then we say that the limit for F is "computed pointwise".

PROPOSITION 4.5.1

Suppose $F\colon \mathbb{I} \longrightarrow [\mathbb{C}, \mathcal{D}]$ is such that for all $C \in \mathbb{C}$,

$$\begin{aligned} F_C\colon \mathbb{I} &\longrightarrow \mathcal{D} \\ I &\longmapsto FI(C) \end{aligned}$$

has a limit cone

$$\left(\textstyle\int_I FI(C) \xrightarrow{(p^C)_I} FI(C)\right)_{I\in\mathbb{I}}.$$

Then F has a limit

$$\left(\textstyle\int_I FI \xrightarrow{k_I} FI\right)_{I\in\mathbb{I}}$$

computed pointwise; i.e.

$$\begin{aligned} \left(\textstyle\int_I FI\right)(C) &= \textstyle\int_I FI(C) \\ \text{and } (k_I)_C &= (p^C)_I \end{aligned}$$

PROOF

We have a functor

$$\begin{aligned} \overline{F}\colon \mathbb{I} \times \mathbb{C} &\longrightarrow \mathcal{D} \\ (I, C) &\longmapsto FI(C) \end{aligned}$$

and each $\overline{F}(\_, C) = F_C$ has a limit, so by parametrized limits, we get a functor

$$C \longmapsto \textstyle\int_I FI(C)$$

Call it L, and claim this gives the limit as required. So we need to show

$$[\mathbb{C}, \mathcal{D}](Y, L) \cong [\mathbb{I}, [\mathbb{C}, \mathcal{D}]](\Delta Y, F)$$

naturally in Y, and to check projections.

Now,

$$\begin{aligned} [\mathbb{C}, \mathcal{D}](Y, L) &\cong \textstyle\int_C \mathcal{D}(YC, LC) && \text{set of nat trans is end in } \mathbf{Set} \\ &= \textstyle\int_C \mathcal{D}(YC, \int_I \overline{F}(I, C)) && \text{rewriting } LC \\ &\cong \textstyle\int_C [\mathbb{I}, \mathcal{D}](\Delta(YC), \overline{F}(\_, C)) && \text{by definition of limit} \\ &\cong [\mathbb{C}, [\mathbb{I}, \mathcal{D}]](\Delta(Y\bullet), \overline{F}(\_, \bullet)) && \text{end in } \mathbf{Set} \text{ is set of nat trans} \\ &\cong [\mathbb{I}, [\mathbb{C}, \mathcal{D}]](\Delta Y, F) \end{aligned}$$

where the last isomorphism holds since

$$[\mathbb{C}, [\mathbb{I}, \mathcal{D}]] \cong [\mathbb{C} \times \mathbb{I}, \mathcal{D}] \cong [\mathbb{I}, [\mathbb{C}, \mathcal{D}]].$$

Note that each line is natural in Y; and the third line gives the projections as required. □

We have the same result for colimits, ends and coends. However, it may be possible for non-pointwise limits to exist if not all the F_C's have limits.

THEOREM 4.5.2

The Yoneda embedding preserves limits.

PROOF

Consider $\mathbb{I} \xrightarrow{D} \mathbb{C} \xrightarrow{H_\bullet} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. Suppose we have a limit cone for D,

$$(\textstyle\int_I DI \xrightarrow{k_I} DI)_{I \in \mathbb{I}}$$

We need to show that $(\mathbb{C}(\_, \int_I DI) \xrightarrow{H_{k_I}} \mathbb{C}(\_, DI))_{I \in \mathbb{I}}$. is a limit for $H_\bullet \circ D$. By the previous result, it suffices to do this pointwise; so for all $C \in \mathbb{C}$, we need that

$$(\mathbb{C}(C \textstyle\int_I DI) \xrightarrow{k_I \circ \_} \mathbb{C}(C, DI))_{I \in \mathbb{I}}$$

is a limit for $I \mapsto \mathbb{C}(C, DI)$, i.e. $H_C \circ D$. But we have already shown this, since representables preserve limits, and the given cone is just H_C of $(\int_I DI \xrightarrow[k_I]{} DI)_{I \in \mathbb{I}}$. □

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THEOREM 4.5.3 (FUBINI)

Suppose $F\colon \mathbb{I} \times \mathbb{J} \longrightarrow \mathcal{D}$ is such that $F_J\colon \mathbb{I} \longrightarrow \mathcal{D}$ has a limit $\int_I F(I, J)$ for all $J \in \mathbb{J}$. Then we have a functor

$$\textstyle\int_I F(I, \_)\colon J \mapsto \int_I F(I, J)$$

such that

$$\textstyle\int_J \int_I F(I, J) \cong \int_{(I,J)} F(I, J)$$

in the sense that if one exists, then so does the other, and they are isomorphic with corresponding limit cones.

PROOF

The right-hand side is a representation of $[\mathbb{I} \times \mathbb{J}, \mathcal{D}](\Delta\_, F)$; the left-hand side is a representation of $[\mathbb{J}, \mathcal{D}](\Delta\_, \int_I F(I, \_))$. Now,

$$\begin{aligned}
[\mathbb{I} \times \mathbb{J}, \mathcal{D}](\Delta V, F) &\cong [\mathbb{I}, [\mathbb{J}, \mathcal{D}]](\Delta V, F(_, _)) \\
&\cong \textstyle\int_I [\mathbb{J}, \mathcal{D}](\Delta V, F(I, _)) \\
&= [\mathbb{J}, \mathcal{D}](\Delta V, \textstyle\int_I F(I, _)).
\end{aligned}$$

Hence representations give the result. □

COROLLARY 4.5.4

Suppose $F\colon \mathbb{I} \times \mathbb{J} \longrightarrow \mathcal{D}$ such that $\int_I F(I, \_)\colon \mathbb{J} \longrightarrow \mathcal{D}$ and $\int_J F(\_, J)\colon \mathbb{I} \longrightarrow \mathcal{D}$ exist. Then

$$\textstyle\int_J \int_I F(I, J) \cong \int_I \int_J F(I, J)$$

in the same sense as above.

PROOF

Both are isomorphic to $\int_{(I,J)} F(I, J)$. □

Note that also we have colimits, ends and coends commuting with themselves; also (co)ends commute with (co)limits.

THEOREM 4.5.5 (DENSITY)

For $X\colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, we have

$$X(U) \cong \textstyle\int^W \mathbb{C}(U, W) \times X(W),$$

naturally in U.

PROOF

We aim to show that

$$[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](X, Y) \cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](\textstyle\int^W \mathbb{C}(\_, W) \times X(W), Y)$$

and deduce result by above. So:

$$\begin{aligned}
\mathrm{RHS} &\cong \textstyle\int_U [\int^W \mathbb{C}(U, W) \times X(W), Y(U)] && \text{set of nat trans is end in } \mathbf{Set} \\
&\cong \textstyle\int_U \int_W [\mathbb{C}(U, W) \times X(W), Y(U)] && \text{restate definition of colimit} \\
&\cong \textstyle\int_W \int_U [\mathbb{C}(U, W) \times X(W), Y(U)] && \text{Fubini interchange} \\
&\cong \textstyle\int_W \int_U [X(W), [\mathbb{C}(U, W), Y(U)]] && \text{definition of function space} \\
&\cong \textstyle\int_W [X(W), \int_U [\mathbb{C}(U, W), Y(U)]] && \text{restate definition of end} \\
&\cong \textstyle\int_W [X(W), Y(W)] && \text{Yoneda restated} \\
&\cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](X, Y) && \text{end in } \mathbf{Set} \text{ is set of nat trans}
\end{aligned}$$

Hence, since the Yoneda embedding is full and faithful, we have the desired natural isomorphism

$$X \cong \textstyle\int^W \mathbb{C}(\_, W) \times X(W). \qquad \square$$

THEOREM 4.5.6

Every presheaf is a colimit of representables.

PROOF

By previous result, we have

$$XU \cong \textstyle\int^{W \in \mathbb{C}} \mathbb{C}(U, W) \times X(W)$$

The idea of the proof is that this is almost a colimit of representables. We would like to say that it is $\int^{W \in \mathbb{C}, x \in X(W)} \mathbb{C}(U, W)$. Can we do this in any way?

We can, by defining the *Grothendieck Fibration*. Given $X \colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, we define a category $\mathbb{G}(X)$ with

- objects being pairs (W, x), $W \in \mathcal{C}$, $x \in XW$.
- morphisms $(W, x) \longrightarrow (W', x')$ being $f \colon W \longrightarrow W'$ such that $Xf(x') = x$.

There is a forgetful functor

$$\begin{aligned}
P \colon \mathbb{G}(X) &\longrightarrow \mathbb{C} \\
(W, x) &\longmapsto W
\end{aligned}$$

So we get $\mathbb{G}(X) \xrightarrow{P} \mathbb{C} \xrightarrow{H_\bullet} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$, and

$$X(U) \cong \textstyle\int^{\alpha \in G(X)} \mathbb{C}(U, P(\alpha))$$

Hence we get $X \cong \int^{\alpha \in G(X)} H_{P(\alpha)}$, a colimit of representables. $\square$

THEOREM 4.5.7

A presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is Cartesian closed.

PROOF

Limits and colimits are computed pointwise, so we get the terminal object and binary products from those in **Set**. So we need to find function spaces. So, given $Y, Z \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$, we seek $Z^Y \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ such that

$$[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](X, Z^Y) \cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](X \times Y, Z)$$

naturally in X and Y. So put

$$\begin{aligned} Z^Y(U) &= [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](H_U \times Y, Z) \\ &\cong \textstyle\int_V [\mathbb{C}(V, U) \times Y(V), Z(V)] && \text{end in } \mathbf{Set}\text{, products ptwise.} \end{aligned}$$

Then

$$\begin{aligned} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](X, Z^Y) &\cong \textstyle\int_U [X(U), Z^Y(U)] && \text{end in } \mathbf{Set} \\ &\cong \textstyle\int_U [X(U), \int_V [\mathbb{C}(V, U) \times Y(V), Z(V)]] && \text{write in definition} \\ &\cong \textstyle\int_U \int_V [X(U), [\mathbb{C}(V, U) \times Y(V), Z(V)]] && \text{restate defn of limit} \\ &\cong \textstyle\int_V \int_U [X(U), [\mathbb{C}(V, U)[Y(V), Z(V)]]] && \text{c.c. of } \mathbf{Set}\text{, Fubini} \\ &\cong \textstyle\int_V \int_U [X(U) \times \mathbb{C}(V, U), [Y(V), Z(V)]] && \text{c.c. of } \mathbf{Set} \\ &\cong \textstyle\int_V [\int^U X(U) \times \mathbb{C}(V, U), [Y(V), Z(V)]] && \text{restate defn of colimit} \\ &\cong \textstyle\int_V [X(V), [Y(V), Z(V)]] && \text{Density} \\ &\cong \textstyle\int_V [X(V) \times Y(V), Z(V)] && \text{c.c. of } \mathbf{Set} \\ &\cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](X \times Y, Z) && \text{end in } \mathbf{Set}\text{, products ptwise.} \end{aligned}$$

Thus Z^Y is a function space as required. □

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5 · Adjunctions

5.1 · *Definitions*

DEFINITION 5.1.1

Let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ be functors. An *adjunction* $F \dashv G$ consists of an isomorphism

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$

that is natural in X and Y. We say F is *left adjoint* to G, and G is *right adjoint* to F.

So, we have a correspondence

$$\begin{matrix} \text{morphisms} \\ FX \to Y \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \text{morphisms} \\ X \to GY \end{matrix}$$

NOTATION

We write

$$\frac{FX \xrightarrow{g} Y \quad \in \mathcal{D}}{X \xrightarrow{\overline{g}} GY \quad \in \mathcal{C}} \qquad \text{and} \qquad \frac{X \xrightarrow{f} GY \quad \in \mathcal{C}}{FX \xrightarrow{\overline{f}} Y \quad \in \mathcal{D}}$$

We write $(\overline{\ })$ for the adjunction operation, and call it transpose. Note $\overline{\overline{f}} = f$, $\overline{\overline{g}} = g$.

What do the naturality conditions mean? Naturality in X says that, for any $h\colon X' \longrightarrow X$,

$$\begin{tikzcd}
\mathcal{D}(FX, Y) \arrow[r, "\overline{(\;)}"] \arrow[d, "_\circ Fh"'] & \mathcal{C}(X, GY) \arrow[d, "_\circ h"] \\
\mathcal{D}(FX', Y) \arrow[r, "\overline{(\;)}"] & \mathcal{C}(X', GY)
\end{tikzcd}$$

commutes. Similarly, naturality in Y says that for any $k\colon Y \longrightarrow Y'$,

$$\begin{tikzcd}
\mathcal{D}(FX, Y) \arrow[r, "\overline{(\;)}"] \arrow[d, "k\circ_"'] & \mathcal{C}(X, GY) \arrow[d, "Gk\circ_"] \\
\mathcal{D}(FX, Y') \arrow[r, "\overline{(\;)}"] & \mathcal{C}(X', GY)
\end{tikzcd}$$

commutes. That is,

$$\frac{X' \xrightarrow{h} X \xrightarrow{f} GY}{FX' \xrightarrow{Fh} FX \xrightarrow{\overline{f}} Y} \quad \text{and} \quad \frac{FX \xrightarrow{g} Y \xrightarrow{k} Y'}{X \xrightarrow{\overline{g}} GY \xrightarrow{Gk} GY'}$$

$$\overline{f \circ h} = \overline{f} \circ Fh \qquad\qquad \overline{k \circ g} = Gk \circ \overline{g}$$

Now, this is actually the Yoneda lemma in disguise:

$$\begin{aligned}
\mathcal{D}(FX, Y) &\cong \mathcal{C}(X, GY) \\
\text{is } H^{FX} &\cong \mathcal{C}(X, G_\,) \\
\text{and } \mathcal{C}(X, GY) &\cong \mathcal{D}(FX, Y) \\
\text{is } H_{GY} &\cong D(F_\,, Y)
\end{aligned}$$

Yoneda tells us that each of these natural transforms is completely determined by where the identity goes:

$$\frac{FX \xrightarrow{1_{FX}} FX}{X \xrightarrow{\eta_X} GFX} \quad \text{and} \quad \frac{GX \xrightarrow{1_{GY}} GY}{FGY \xrightarrow{\varepsilon_Y} Y}$$

Then by naturality,

$$\overline{g} = Gg \circ \eta_X \qquad \begin{matrix} FX \xrightarrow{1_{FX}} FX \xrightarrow{g} Y \\ X \xrightarrow{\eta_X} GFX \xrightarrow{Gg} GY \end{matrix}$$

and

$$\overline{f} = \varepsilon_X \circ Ff \qquad \begin{matrix} X \xrightarrow{f} GY \xrightarrow{1_{GY}} GY \\ FX \xrightarrow{Ff} FGY \xrightarrow{\varepsilon_Y} Y \end{matrix}$$

And in fact, the η_X, ε_Y are components of a natural transformation.

PROPOSITION 5.1.2

Given $F \dashv G$, we have natural transformations η and ε with components given by η_X, ε_Y.

PROOF

Check naturality. For η, given $f\colon X \longrightarrow X'$,

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & GFX \arrow[d, "GFf"] \\ X' \arrow[r, "\eta_{X'}"'] & GFX' \end{tikzcd}$$

must commute. Now, we have:-

$$\begin{array}{cccccc} X & \xrightarrow{\eta_X} & GFX & \xrightarrow{GFf} & GFX' & \\ \hline FX & \xrightarrow{1_{FX}} & FX & \xrightarrow{Ff} & FX' & \xrightarrow{1_{FX'}} \; FX' \\ \hline & & X & \xrightarrow{f} & X' & \xrightarrow{\eta_{X'}} \; GFX' \end{array}$$

But we have transposed twice, and hence we have equality as required. Similarly for ε. $\square$

DEFINITION 5.1.3

Given $F \dashv G$, we call $\eta\colon 1_{\mathcal{C}} \Rightarrow GF$ the *unit* and $\varepsilon\colon FG \Rightarrow 1_{\mathcal{D}}$ the *counit* of the adjunction.

5.2 · *Examples*

EXAMPLES 5.2.1

Free $\dashv$ forgetful. For example:

1 $U\colon \mathbf{Gp} \longrightarrow \mathbf{Set}$ has a left adjoint $F \dashv U$, where $F(S)$ gives the free group on S; so we have

$$\mathbf{Gp}(FS, G) \cong \mathbf{Set}(S, U(G))$$

2 $U\colon \mathbf{Alg} \longrightarrow \mathbf{Vect}$ which forgets the multiplicative structure; we have $F \dashv U$, where $F(V)$ is the free algebra on V.
3 $U\colon \mathbf{Ring} \longrightarrow \mathbf{Monoid}$ has a left adjoint

$$\mathbb{Z} \circ \_ : M \longmapsto \mathbb{Z}M = \{\text{formal finite combinations } \textstyle\sum \lambda_i m_i, \lambda_i \in \mathbb{Z}, m_i \in M.\}$$

4 $U\colon \mathbf{Ab} \longrightarrow \mathbf{Gp}$ has a left adjoint "free abelianization": $G^{\mathrm{AB}} = G/[G, G]$.
5 $U\colon \mathbf{Alg}_k \longrightarrow \mathbf{Lie}_k$ has left adjoint $L \mapsto \mathcal{U}(L) =$ universal enveloping algebra of L.

EXAMPLES 5.2.2

Reflections $\dashv$ inclusions $\dashv$ coreflections. If $\mathcal{C} \longrightarrow \mathcal{D}$ has a left adjoint, it is called a *reflector* and exhibits $\mathcal{C}$ as a *reflective subset* of $\mathcal{D}$.

1 As above, $\mathbf{Ab} \longrightarrow \mathbf{Gp}$; $\mathbf{Ab}$ is reflective in $\mathbf{Gp}$.
2

$$\left\{ \begin{array}{c} \text{complete metric spaces,} \\ \text{uniformly cts functions} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{metric spaces,} \\ \text{uniformly cts functions} \end{array} \right\}$$

has left adjoint "completion".

3

$$\left\{ \begin{array}{c} \text{compact Hausdorff spaces,} \\ \text{uniformly cts functions} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{topological spaces,} \\ \text{uniformly cts functions} \end{array} \right\}$$

has left adjoint Stone-Čech compactification.

4 **Gp** $\longrightarrow$ **Monoid**. **Gp** is reflective and coreflective in **Monoid**, via

$$M \longmapsto \{\, m \in M \mid m \text{ is invertible} \,\}$$

EXAMPLE 5.2.3

Closedness. Let $\mathcal{C}$ be a cartesian closed category. Then for all $B \in \mathcal{C}$, we have

$$\_ \times B \dashv (\_)^B$$

i.e.

$$\mathcal{C}(A \times B, C) \cong \mathcal{C}(A, C^B)$$

naturally in A and C.

EXAMPLE 5.2.4

Adjoints for representable functors are powers and copowers. Recall given an object $A \in \mathcal{C}$ and a set I, we can form the *I-fold power*:

$$A^I = \prod_{i \in I} A = [I, A]$$

and dually the *I-fold copower*:

$$I \times A = \coprod_{i \in I} A.$$

By parametrised limits, we get functors:

$$[\_, A] \colon \mathbf{Set} \longrightarrow \mathcal{C}^{\mathrm{op}}$$
$$\_ \times A \colon \mathbf{Set} \longrightarrow \mathcal{C}$$

Now, $\mathbf{Set}(I, \mathcal{C}(U, A)) \cong \mathcal{C}(U, [I, A]) \cong \mathcal{C}^{\mathrm{op}}([I, A], U)$. So $[\_, A] \dashv \mathcal{C}(\_, A) = H_A$. Similarly $\_ \times A \dashv \mathcal{C}(A, \_) = H^A$, since $\mathbf{Set}(I, \mathcal{C}(A, U)) \cong \mathcal{C}(I \times A, U)$.

So H_A has an adjoint iff $\mathcal{C}$ has all small powers of A iff $\mathcal{C}^{\mathrm{op}}$ has all small copowers of A.

If $\mathcal{C}$ has all small powers and copowers of A, we get

$$\mathcal{C}(I \times A, U) \cong \mathcal{C}(A, [I, U])$$

via $\mathbf{Set}(I, \mathcal{C}(A, U))$. So $I \times \_ \dashv [I, \_] \colon \mathcal{C} \longrightarrow \mathcal{C}$.

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5.3 · *Triangle identities*

PROPOSITION 5.3.1

Given an adjunction $F \dashv G$, then the unit $\eta \colon 1 \Rightarrow GF$ and the counit $\varepsilon \colon FG \Rightarrow 1$ satisfy the triangle identities; that is, the following diagrams commute:

$$\begin{tikzcd} GY \arrow[r, "\eta_{GY}"] \arrow[dr, "1_{GY}"'] & GFGY \arrow[d, "G\varepsilon_Y"] \\ & GY \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FX \arrow[r, "F\eta_X"] \arrow[dr, "1_{FX}"'] & FGFX \arrow[d, "\varepsilon_X"] \\ & FX \end{tikzcd}$$

PROOF

$$
\begin{array}{ccccc}
GY & \xrightarrow{\eta_{GY}} & GFGY & \xrightarrow{G\varepsilon_Y} & GY \\
\hline
FGY & \xrightarrow{1_{FGY}} & FGY & \xrightarrow{\varepsilon_Y} & Y \\
\cline{3-5}
 & & GY & \xrightarrow{1_{GY}} & GY
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
FX & \xrightarrow{F\eta_X} & FGFX & \xrightarrow{\varepsilon_X} & FX \\
\hline
X & \xrightarrow{\eta_X} & GFX & \xrightarrow{1_{GFX}} & GFX \\
\cline{1-3}
FX & \xrightarrow{1_{FX}} & FX & &
\end{array}
$$

□

THEOREM 5.3.2

An adjunction $F \dashv G$ is completely determined by natural transformations

$$\eta\colon 1 \Rightarrow GF$$
$$\varepsilon\colon FG \Rightarrow 1$$

satisfying the triangle identities.

PROOF

Suppose we are given such ε, η. We need to show that

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$$

naturally in X and Y. So, given $f\colon X \longrightarrow GY$, put

$$\overline{f}\colon FX \xrightarrow{Ff} FGY \xrightarrow{\varepsilon_Y} Y$$

and given $g\colon FX \longrightarrow Y$, put

$$\overline{g}\colon X \xrightarrow{\eta_X} GFX \xrightarrow{Gg} GY$$

We need to check naturality. For naturality in X, we need, given $h\colon X' \longrightarrow X$, that $\overline{fh} = \overline{f} \circ Fh$. Now,

$$
\begin{aligned}
\overline{fh} &= \varepsilon_Y \circ F(fh) \\
&= (\varepsilon_Y \circ Ff) \circ Fh \\
&= \overline{f} \circ Fh.
\end{aligned}
$$

For naturality in Y, we need, for all $k\colon Y \longrightarrow Y'$, $\overline{kg} = Gk \circ \overline{g}$. Now,

$$
\begin{aligned}
\overline{kg} &= G(kg) \circ \eta_Y \\
&= Gk \circ (Gg \circ \eta_Y) \\
&= Gk \circ \overline{g}.
\end{aligned}
$$

Now we need to check that these are inverse: given $f\colon X \longrightarrow GY$, we need that $f = \overline{\overline{f}}$. We have

$$\overline{f} = FX \xrightarrow{Ff} FGY \xrightarrow{\varepsilon_Y} Y.$$

So

$$
\begin{tikzcd}
\overline{\overline{f}} = X \arrow[r, "\eta_X"] \arrow[drr, "f"'] & GFX \arrow[r, "GFf"] & GFGY \arrow[r, "G\varepsilon_Y"] & GY \\
 & & GY \arrow[u, "\eta_{GY}"'] \arrow[ur, "1_{GY}"'] &
\end{tikzcd}
$$

Note that the left hand circuit commutes by the naturality of η, and the right hand circuit commutes by the first triangle identity, so $f = \overline{\overline{f}}$. Similarly, given $g\colon FX \longrightarrow Y$,

$$\overline{\overline{g}} = \begin{tikzcd} FX \arrow[r, "F\eta_X"] \arrow[dr, "1_{FX}"'] & FGFX \arrow[r, "FGg"] \arrow[d, "\varepsilon_{FX}"] & FGY \arrow[r, "\varepsilon_Y"] & Y \\ & FX \arrow[urr, "g"'] & & \end{tikzcd}$$

Here, the left circuit commutes by the second triangle identity, and the right circuit commutes by the naturality of ε; hence $g = \overline{\overline{g}}$, as required. $\square$

REMARK

Adjunctions can be composed:

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightleftarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightleftarrows}} \mathcal{E} \qquad \text{giving} \qquad \mathcal{C} \underset{G_1G_2}{\overset{F_2F_1}{\rightleftarrows}} \mathcal{E}$$

from $\mathcal{E}(F_2F_1X, Y) \cong \mathcal{D}(F_1X, G_2Y) \cong \mathcal{C}(X, G_1G_2Y)$.

5.4 · *Adjunctions as parametrised representations*

To give a left adjoint to $G\colon \mathcal{D} \longrightarrow \mathcal{C}$, it is sufficient to give, for each $X \in \mathcal{C}$, a representation for

$$\mathcal{C}(X, G\_)\colon \mathcal{D} \longrightarrow \mathbf{Set}.$$

By parametrised representation, this extends uniquely to a functor which is the left adjoint we are looking for. Dually, a right adjoint to $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a representation for

$$\mathcal{D}(F\_, Y)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

Recall "$\mathcal{D}$ has limits of shape $\mathbb{I}$" means, for all $D\colon \mathbb{I} \longrightarrow \mathcal{D}$, there exists a representation of

$$[\mathbb{I}, \mathcal{D}](\Delta\_, D)\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

i.e., $\mathcal{D}$ has limits of shape $\mathbb{I}$ iff $\Delta\_ : \mathcal{D} \longrightarrow [\mathbb{I}, \mathcal{D}]$ has a right adjoint. Dually, $\mathcal{D}$ has colimits of shape $\mathbb{I}$ iff $\Delta\_ : \mathcal{D} \longrightarrow [\mathbb{I}, \mathcal{D}]$ has a left adjoint.

5.5 · *Adjunctions as collections of initial objects*

DEFINITION 5.5.1

Given $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ and $X \in \mathcal{C}$, we define the *comma category* $(X \downarrow G)$:

- objects are pairs (f, Y), $X \xrightarrow{f} GY$;
- morphisms $(f, Y) \xrightarrow[h]{} (f', Y')$ are morphisms $Y \xrightarrow[h]{} Y'$ such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ GY \arrow[rr, "Gh"'] & & GY' \end{tikzcd}$$

commutes.

PROPOSITION 5.5.2

To give a left adjoint for $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ is equivalent to giving, for all $X \in \mathcal{C}$, an initial object for the comma category $(X \downarrow G)$.

PROOF

An initial object in $(X \downarrow G)$ is a pair (u, V_X) with $X \xrightarrow{u} GV_X$ such that, for all $X \xrightarrow{f} GY$, there exists a unique $h\colon V_X \longrightarrow Y$ such that

$$\begin{tikzcd} & X \arrow[dl, "u"'] \arrow[dr, "f"] & \\ GV_X \arrow[rr, "Gh"'] & & GY' \end{tikzcd}$$

commutes. So

$$\begin{aligned} \mathcal{D}(V_X, Y) &\cong \mathcal{C}(X, GY) \\ f &\mapsto Gh \circ u \end{aligned}$$

We need to check naturality in Y. So, for all $g\colon Y \longrightarrow Y'$, we have

$$\begin{tikzcd} \mathcal{D}(V_X, Y) \arrow[r] \arrow[d] & \mathcal{C}(X, GY) \arrow[d] \\ \mathcal{D}(V_X, Y') \arrow[r] & \mathcal{C}(X, GY') \end{tikzcd}$$

and so on elements

$$\begin{tikzcd} h \arrow[r, mapsto] \arrow[d, mapsto] & Gh \circ u \arrow[d, mapsto] \\ g \circ h \arrow[r, mapsto] & G(g \circ h) \circ u = Gg \circ Gh \circ u \end{tikzcd}$$

and so this is a representation as required. □

5.6 · *Duality*

We note that there are a lot of duality relations going on with adjunctions:

left adjoint	$\longleftrightarrow$	right adjoint
unit	$\longleftrightarrow$	counit
natural in X	$\longleftrightarrow$	natural in Y
first triangle identity	$\longleftrightarrow$	second triangle identity

Why is this? Consider

$$F \dashv G,\ \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad \text{also} \qquad G \dashv F,\ \mathcal{C}^{\mathrm{op}} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}^{\mathrm{op}}$$

$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$	$\mathcal{D}^{\mathrm{op}}(Y, FX) \cong \mathcal{C}^{\mathrm{op}}(GY, X)$
$F \dashv G\colon \mathcal{D} \longrightarrow \mathcal{C}$	$G \dashv F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{op}}$
unit $\eta_X\colon X \longrightarrow GFX$	counit $\eta_X\colon GFX \longrightarrow X$
counit $\varepsilon_Y\colon FGY \longrightarrow Y$	unit $\varepsilon_Y\colon Y \longrightarrow FGY$

6 · Adjoint functor theorems

6.1 · *Preservation*

THEOREM 6.1.1

Suppose $F \dashv G\colon \mathcal{D} \longrightarrow \mathcal{C}$. Then G preserves limits, and F preserves colimits.

PROOF

Consider $D\colon \mathbb{I} \longrightarrow \mathcal{D}$ with limit cone $(\int_I DI \xrightarrow{k_I} DI)_{I\in\mathbb{I}}$. We need to show that G of it is a limit cone for $GD\colon \mathbb{I} \longrightarrow \mathcal{C}$. The cone becomes

$$(G \textstyle\int_I DI \xrightarrow{Gk_I} GDI)_{I\in\mathbb{I}}.$$

We need a natural transformations $\mathcal{C}(\_, G \int_I DI) \cong [\mathbb{I}, \mathcal{C}](\Delta\_, GD)$ with components

$$\begin{aligned} \mathcal{C}(V, G \textstyle\int_I DI) &\cong [\mathbb{I}, \mathcal{C}](\Delta V, GD) \\ f &\mapsto (Gk_I \circ f)_{I\in\mathbb{I}} \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{C}(V, G \textstyle\int_I DI) &\cong \mathcal{D}(FV, \textstyle\int_I DI) \\ &\cong \textstyle\int_I \mathcal{D}(FV, DI) \\ &\cong \textstyle\int_I \mathcal{C}(V, GDI) \\ &\cong [\mathbb{I}, \mathcal{C}](\Delta V, GD). \end{aligned}$$

And on projections:

$$\begin{aligned} f &\mapsto \overline{f} \\ &\mapsto k_I \circ \overline{f} \\ &\mapsto Gk_I \circ f \end{aligned}$$

as required; and dually for F. □

6.2 · *General adjoint functor theorem*

DEFINITION 6.2.1

Given a category $\mathcal{A}$, a collection $\mathbb{I} \subseteq \mathcal{A}$ is *weakly initial* if for all $A \in \mathcal{A}$, there exists a morphism $I \longrightarrow A$ for some $I \in \mathbb{I}$.

EXAMPLE

{initial object} is a weakly initial set.

THEOREM 6.2.2 (GENERAL ADJOINT FUNCTOR THEOREM)

Suppose we have a functor $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ that preserves small limits, and that $\mathcal{D}$ is locally small and complete. Then G has a left adjoint iff for all $X \in \mathcal{C}$, the category $(X \downarrow G)$ has a weakly initial set.

This last condition is known as the *solution set condition*.

PROOF

Here is the general structure of the proof:

$$\begin{tikzcd}[column sep=small]
\mathcal{D}\text{ locally small} \arrow[d] & & \mathcal{D}\text{ complete} \arrow[ddr] & & \begin{array}{c}G\text{ preserves}\\ \text{small limits}\end{array} \arrow[d] \\
\begin{array}{c}(X \downarrow G)\text{ locally}\\ \text{small}\end{array} \arrow[ddr] & & & & \begin{array}{c}P\colon (X \downarrow G)\\ \longrightarrow \mathcal{D}\text{ creates}\\ \text{small limits}\end{array} \arrow[dl] \\
& & & \begin{array}{c}(X \downarrow G)\\ \text{complete}\end{array} \arrow[dll] & \\
& \begin{array}{c}(X \downarrow G)\text{ has}\\ \text{w.i.s. iff}\\ (X \downarrow G)\text{ has}\\ \text{initial object}\end{array} \arrow[dr] & & & \begin{array}{c}G\text{ has a left}\\ \text{adjoint iff for}\\ \text{all }X, (X \downarrow G)\\ \text{has initial}\\ \text{object}\end{array} \arrow[dll] \\
& & \text{GAFT} & &
\end{tikzcd}$$

where we define $P\colon (X \downarrow G) \longrightarrow \mathcal{D}$ to be the obvious forgetful functor. So:

LEMMA 1

$P\colon (X \downarrow G) \longrightarrow \mathcal{D}$ creates small limits.

PROOF

Let $D\colon \mathbb{I} \longrightarrow (X \downarrow G)$ be a diagram. We need to show that, if PD has a limit cone, then there is a cone

$$(V \xrightarrow{c_I} DI)_{I \in \mathbb{I}}$$

in $(X \downarrow G)$ such that $(PV \xrightarrow{Pc_I} PDI)_{I \in \mathbb{I}}$ is a limit for PD in $\mathcal{D}$, and that any such cone is itself a limit for D in $(X \downarrow G)$.

1 Suppose $PD\colon \mathbb{I} \longrightarrow \mathcal{D}$ has a limit cone, say $(L \xrightarrow{c_I} PDI)_{I \in \mathbb{I}}$:

$$\begin{tikzcd}
& L \arrow[dl, "c_I"'] \arrow[d, "c_{I'}"] \arrow[dr, "c_{I''}"] & \\
PDI \arrow[r, shift left] \arrow[r, shift right] & PDI' \arrow[r, shift left] \arrow[r, shift right] & PDI''
\end{tikzcd}$$

2 G preserves small limits, so $(GL \xrightarrow{Gc_I} GPDI)_{I \in \mathbb{I}}$ is a limit for GPD in $\mathcal{C}$.

$$\begin{tikzcd}
& GL \arrow[dl, "Gc_I"'] \arrow[d, "Gc_{I'}"] \arrow[dr, "Gc_{I''}"] & \\
GPDI \arrow[r, shift left] \arrow[r, shift right] & GPDI' \arrow[r, shift left] \arrow[r, shift right] & GPDI''
\end{tikzcd}$$

3 $(DI)_{I \in \mathbb{I}}$ gives a diagram in $(X \downarrow G)$

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[d] \arrow[dr] & \\
GPDI \arrow[r, shift left] \arrow[r, shift right] & GPDI' \arrow[r, shift left] \arrow[r, shift right] & GPDI''
\end{tikzcd}$$

which is precisely a cone $(X \longrightarrow GPDI)_{I \in \mathbb{I}}$ in $\mathcal{C}$. Hence we induce a unique morphism $u \colon X \longrightarrow GL$ making everything commute:

$$\begin{tikzcd}
 & X \arrow[rr, dotted] \arrow[dl] \arrow[d] \arrow[drr] & & GL \arrow[dlll] \arrow[dll] \arrow[dr] & \\
GPDI \arrow[r, shift left] \arrow[r, shift right] & GPDI' \arrow[rr, shift left] \arrow[rr, shift right] & & & GPDI''
\end{tikzcd} \qquad (1)$$

4 Since everything in the diagram commutes, it forms a cone over D in $(X \downarrow G)$, with vertex $V = (X \xrightarrow{u} GL)$. Moreover, by construction is it unique such that applying P to it gives the original cone $(L \xrightarrow{c_I} PDI)_{I \in \mathbb{I}}$. So we have shown that, given a limit cone for PD there is a unique cone in $(X \downarrow G)$ that maps to it, given by (1) above. It remains to show that this cone is universal.

5 Given any cone $((X \xrightarrow{f} GY) \longrightarrow DI)_{I \in \mathbb{I}}$ in $(X \downarrow G)$, we seek a unique factorisation $(X \xrightarrow{f} GY) \longrightarrow V$:

$$\begin{tikzcd}
 & GY \arrow[ddr] \arrow[dd] \arrow[drr, dotted, "Gh"] & & \\
X \arrow[ur, "f"] \arrow[rrr] \arrow[d] \arrow[dr] & & & GL \arrow[dll] \arrow[dl] \\
GPDI \arrow[r, shift left] \arrow[r, shift right] & GPDI' & &
\end{tikzcd}$$

Applying P, we get a cone $(Y \longrightarrow PDI)_{I \in \mathbb{I}}$ in $\mathcal{D}$, and since L is a limit, this induces a unique morphism $h \colon Y \longrightarrow L$ making everything commute in $\mathcal{D}$. But now, by the uniqueness of u we have $Gh \circ f = u$, since $Gh \circ f$ satisfies the conditions making u unique. So h is a morphism in $(X \downarrow G)$:

$$\begin{tikzcd}
 & GY \arrow[dr, "Gh"] & \\
X \arrow[ur, "f"] \arrow[rr, "u"'] & & GL
\end{tikzcd}$$

and so is the unique factorisation as required. So the cone (1) is indeed universal and P creates limits as required. □

So now we can quickly deduce

LEMMA 2

For each $X \in \mathcal{C}$, $(X \downarrow G)$ is locally small and complete.

PROOF

Since $\mathcal{D}$ is locally small, so too is $(X \downarrow G)$. Now, let D be a diagram in $(X \downarrow G)$. Apply P to get a diagram PD in $\mathcal{D}$. This has a limit, since $\mathcal{D}$ is complete. And by lemma 1, P creates it from a limit in $(X \downarrow G)$; i.e. D has a limit in $(X \downarrow G)$. So $(X \downarrow G)$ is complete. □

Now, we need only prove

LEMMA 3 (INITIAL OBJECT LEMMA)

If $\mathcal{A}$ is locally small and complete, then $\mathcal{A}$ has an initial object iff $\mathcal{A}$ has a weakly initial set.

PROOF

$\Rightarrow$ is clear; so we need to show $\Leftarrow$. So let $\mathbb{I}$ be a weakly initial set in $\mathcal{A}$. We need to construct an initial object from $\mathbb{I}$.

So, set $P = \prod_{I \in \mathbb{I}} I$. This is a small product, since $\mathbb{I}$ is a set. Now set L to be a limit over the diagram of all morphisms $P \rightrightarrows P$; this is a small limit since $\mathcal{A}$ is locally small. We claim that L is initial in $\mathcal{A}$. Note that L has projections

$$\begin{tikzcd} & L \arrow[dl, "k"'] \arrow[dr, "k'"] & \\ P \arrow[rr, shift left=2] \arrow[rr] \arrow[rr, shift right=2] & & P \end{tikzcd}$$

Now:

1 $k = k'$ since all triangles commute, and we have $1_P \colon P \longrightarrow P$;
2 for all $f \colon P \longrightarrow P$, $fk = k$, since all triangles commute;
3 k is monic (c.f. proof that an equaliser is monic).

We immediately have that I weakly initial $\Rightarrow$ $\{P\}$ weakly initial $\Rightarrow$ $\{L\}$ weakly initial. So for all $A \in \mathcal{A}$, there exists a morphism $L \longrightarrow A$.

We need to show this morphism is unique. So suppose we have $L \overset{s}{\underset{t}{\rightrightarrows}} A$. Consider

$$\begin{tikzcd} & & & & P \\ L \arrow[r, "k"] & P \arrow[r, "m"] & E \arrow[r, "e"] & L \arrow[ur, "k"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A \end{tikzcd}$$

where $E \xrightarrow{e} L$ is an equaliser of s and t.

Now, $(kem)k = k$ by (1) above. But k is monic, and $k(emk) = k \circ 1$, so $emk = 1$. Now $se = te$ since e is an equaliser. Hence

$$s = semk = temk = t$$

as required. So L is indeed an initial object. □

So now by lemmas 2 and 3 together with Proposition 5.5.2, we deduce that G has a left adjoint iff, for each $X \in \mathcal{C}$, $(X \downarrow G)$ has a weakly initial set, as required. □

LECTURE 18 · 20/11/02

6.3 · *Special adjoint functor theorem*

DEFINITION 6.3.1

Consider monics $A \rightarrowtail X$. Define $a \leqslant b$ iff $\exists c \colon A \longrightarrow B$ such that

$$\begin{tikzcd} A \arrow[rr, "c"] \arrow[dr, tail, "a"'] & & B \arrow[dl, tail, "b"] \\ & X & \end{tikzcd}$$

commutes. Observe that if there exists such a c, then it is unique (since b is monic) and monic (since a is monic). Now, set $a \sim b$ iff $a \leqslant b$ and $b \leqslant a$. The equivalence classes under $\sim$ are called *subobjects* of X.

DEFINITION 6.3.2

A category $\mathcal{C}$ is *wellpowered* iff for all $X \in \mathcal{C}$, the collection of subobjects of X is a set; equivalently, iff there exists a set of representing monics into X.

DEFINITION 6.3.3

A collection $\mathbb{B} \longrightarrow \mathcal{D}$ is *cogenerating* if whenever $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ such that

$$\forall Y \xrightarrow{b} B,\ B \in \mathbb{B},\ bf = bg$$

then $f = g$.

THEOREM 6.3.4 (SPECIAL ADJOINT FUNCTOR THEOREM)

Suppose $G\colon \mathcal{D} \longrightarrow \mathcal{C}$ such that

- $\mathcal{C}$ is locally small;
- $\mathcal{D}$ is locally small, complete, well-powered and has a cogenerating set;

Then G has a left adjoint iff it preserves limits.

PROOF

$\Rightarrow$ is clear; the point is $\Leftarrow$. We aim to show that each $(X \downarrow G)$ has a weakly initial set, so we can apply GAFT. That is, given any $X \in \mathcal{C}$, we find a set $\mathbb{A} \subseteq (X \downarrow G)$ such that for each $f\colon X \longrightarrow GY \in (X \downarrow G)$, there exists morphism

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "f"'] & GA \arrow[d, "Gk"] \\ & GY \end{tikzcd}$$

for some $X \xrightarrow{a} GA \in \mathbb{A}$. So we fix X and construct such a set $\mathbb{A}$. Let $\mathbb{B}$ be a cogenerating set in $\mathcal{D}$.

1 Put

$$Q_X = \prod_{\substack{X \xrightarrow{x} GB, \\ B \in \mathbb{B}}} B$$

with projections $Q_X \xrightarrow{q_x} B$ (one for each $X \xrightarrow{x} GB$). This is a small product since $\mathbb{B}$ is a set and $\mathcal{C}$ is locally small.

$$\begin{tikzcd} & Q_X \arrow[dl, "q_x"'] \arrow[d, "q_{x'}"] \arrow[dr] & \\ B & B' & \dots \end{tikzcd}$$

2 $\mathcal{D}$ is well-powered, so pick a set of representing monics into Q_X (i.e. one monic for each

isomorphism class). Write $\mathbb{M} = \{\text{representing monics } A \rightarrowtail Q\}$.

$$\begin{tikzcd} A \arrow[dr, tail, "m"'] & A' \arrow[d, tail, "m'"] & \cdots \\ & Q_X & \end{tikzcd}$$

3 Put

$$\mathbb{A} = \{X \xrightarrow{a} GA \text{ such that } \exists A \overset{m}{\rightarrowtail} Q_X \in \mathbb{M}\} \subseteq (X \downarrow G).$$

This is a set since $\mathbb{M}$ is a set and $\mathcal{C}$ is locally small. We claim that $\mathbb{A}$ is the desired weakly initial set in $(X \downarrow G)$. So we need to show, given any $f\colon X \longrightarrow GY \in (X \downarrow G)$, that there exists

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "f"'] & GA \arrow[d, "Gt"] \\ & GY \end{tikzcd}$$

with $X \xrightarrow{a} GA \in \mathbb{A}$. So we fix $X \xrightarrow{f} GY$ and seek such a triangle.

4 Put

$$P_Y = \prod_{\substack{y\colon Y \longrightarrow B \\ B \in \mathbb{B}}} B$$

with projections $P_Y \xrightarrow{p_y} B$ (one for each $y\colon Y \longrightarrow B$.

$$\begin{tikzcd} & P_Y \arrow[dl, "p_y"'] \arrow[d, "p_{y'}"] \arrow[dr] & \\ B & B' & \cdots \end{tikzcd}$$

AIM

$$\begin{tikzcd} X \arrow[dr, dotted, "a"] \arrow[drr, "f"] \arrow[ddr, "h"'] & & \\ & GA \arrow[r, "Gt"] \arrow[d, tail, "Gg"'] & GY \arrow[d, tail, "Gd"] \\ & GQ_X \arrow[r, "Ge"] & GP_Y \end{tikzcd} \qquad (*)$$

- form $Y \longrightarrow P_Y$, show monic;
- form $Q_X \longrightarrow P_Y$;
- take pullback; G preserves pullbacks;
- form $X \longrightarrow GQ_X$ making outside commute;
- induce $X \xrightarrow{a} GA$ as required;
- $a \in \mathbb{A}$ since g monic.

5 Induce $T \xrightarrow{d} P_Y$ by the universal property of the product P_Y:

$$\begin{tikzcd} & Y \arrow[d, dotted, "d"] \arrow[ddl, "y"'] \arrow[ddr, "y'"] & \\ & P_Y \arrow[dl, "p_y"] \arrow[dr, "p_{y'}"'] & \\ B & \cdots & B' \end{tikzcd}$$

So we get unique d such that

$$\forall y\colon Y \longrightarrow B, \quad p_y \circ d = y \qquad\qquad (1).$$

We show that d is monic; suppose we have $\overset{s}{\underset{t}{\rightrightarrows}} Y \xrightarrow{d} P_Y$ with $ds = dt$. Then certainly, for all $y\colon Y \longrightarrow B$, $p_y ds = p_y dt$. So by (1), for all $y\colon Y \longrightarrow B$, $ys = yt$. Hence $s = t$ since $\mathbb{B}$ is cogenerating. Hence d is monic.

6 Induce $Q_X \xrightarrow{e} P_Y$ by the universal property of product P_Y. To use this, we need to find for each $Y \xrightarrow[y]{} B$ a morphism $Q_X \longrightarrow B$.

Now, we have a projection $Q_X \xrightarrow{q_x} B$ for all $x\colon X \longrightarrow GB$, and given any $Y \xrightarrow{y} B$, we certainly have a morphism

$$x = X \xrightarrow{f} GY \xrightarrow{Gy} GB$$

so we can use projections $q_{Gy\circ f}\colon Q_X \longrightarrow B$:

$$\begin{tikzcd}
& Q_X \arrow[dl, "q_{Gy\circ f}"'] \arrow[d, dotted, "e"] \arrow[dr, "q_{Gy'\circ f}"] & \\
& P_Y \arrow[dl, "p_y"] \arrow[dr, "p_{y'}"'] & \\
B & \cdots & B'
\end{tikzcd}$$

inducing a unique $e\colon Q_X \longrightarrow P_Y$ such that

$$\forall y\colon Y \longrightarrow B, \quad q_{Gy\circ f} = p_y \circ e \qquad\qquad (2).$$

7 Form the pullback

$$\begin{tikzcd}
A \arrow[r, "t"] \arrow[d, tail, "g"'] & Y \arrow[d, tail, "d"] \\
Q_X \arrow[r, "e"] & P_Y
\end{tikzcd}$$

Now d is monic, so g is monic; without loss of generality we can assume g is a representing monic (since it must be isomorphic to one, so we can take an isomorphic pullback). G preserves pullbacks so

$$\begin{tikzcd}
GA \arrow[r] \arrow[d, tail, "Gg"'] & GY \arrow[d, tail, "Gd"] \\
GQ_X \arrow[r, "Ge"] & GP_Y
\end{tikzcd}$$

is also a pullback.

8 Induce $X \xrightarrow{h} GQ_X$ by the universal property of the product GQ_X. Since G preserves limits, GQ_X is indeed a product,

$$GQ_X = \prod_{\substack{X \xrightarrow{x} GB \\ B \in \mathbb{B}}} GB$$

with projections $GQ_X \xrightarrow{Gq_x} GB$, one for each $x\colon X \longrightarrow GB$, $B \in \mathbb{B}$).

$$\begin{tikzcd}
& X \arrow[dl, "x"'] \arrow[d, dotted, "h"] \arrow[dr, "x'"] & \\
& GQ_X \arrow[dl, "Gq_x"] \arrow[dr, "Gq_{x'}"'] & \\
GB & \cdots & GB'
\end{tikzcd}$$

So we have unique h such that

$$\forall x\colon X \longrightarrow GB, \quad Gq_x \circ h = x \qquad\qquad (3).$$

9 We now show that the outside of the diagram (*) commutes, using the universal property of the product GP_Y. For each $y\colon Y \longrightarrow B$, we have the following diagram:

$$\begin{tikzcd}
X \arrow[drr, "f"] \arrow[ddr, "h"'] \arrow[ddrr, dotted, "k"] & & & \\
& & GY \arrow[d, "Gd"'] \arrow[ddr, bend left, "Gy"] & \\
& GQ_X \arrow[r, "Ge"] \arrow[drr, bend right, "Gq_{Gy\circ f}"'] & GP_Y \arrow[dr, "Gp_y"] & (1) \\
& & (3) & GB
\end{tikzcd}$$

Now, the outside commutes by (3), and the triangles commute as shown. So we need show that $Ge \circ h = Gd \circ f$.

AIM

> We use the universal property of the product GP_Y to induce a unique k such that for all $y\colon Y \longrightarrow B$, $Gp_y \circ k = Gy \circ f$; then we show that $Ge \circ h$ and $Gd \circ f$ both satisfy this condition.

10 G preserves limits, so GP_Y is a product

$$GP_Y = \prod_{\substack{y\colon Y \longrightarrow B \\ B \in \mathbb{B}}} GB$$

with projections $GP_Y \xrightarrow{Gp_y} GB$. Now, for each $y\colon Y \longrightarrow B$ we have a morphism $X \xrightarrow{Gg\circ f} GB$:

$$\begin{tikzcd}
& X \arrow[dl, "Gy\circ f"'] \arrow[d, dotted, "k"] \arrow[dr, "Gy'\circ f"] & \\
& GP_Y \arrow[dl, "Gp_y"] \arrow[dr, "Gp_{y'}"'] & \\
GB & \cdots & GB'
\end{tikzcd}$$

inducing a unique $k\colon X \longrightarrow GP_Y$ such that

$$\forall y\colon Y \longrightarrow B, \quad Gp_y \circ k = Gy \circ f \qquad (4).$$

11 $Ge \circ h$ and $Gd \circ f$ both satisfy this condition, since for all $y\colon Y \longrightarrow B$, we have

$$Gp_y \circ Gd \circ f = G(p_y \circ d) \circ f \overset{(1)}{=} Gy \circ f$$

and

$$Gp_y \circ Ge \circ h = G(p_y \circ e) \circ h \overset{(2)}{=} Gq_{Gy \circ f} \circ h \overset{(3)}{=} Gy \circ f.$$

Hence by the uniqueness of k, we have $Ge \circ h = Gd \circ f$ and so the outside of (*) commutes.

12 Induce $X \xrightarrow{a} GA$ by the universal property of pullback (as in (*)). Then $X \xrightarrow{a} GA \in \mathbb{A}$ since there exists monic $A \overset{g}{\rightarrowtail} Q_X \in \mathbb{M}$, and we have a commuting triangle

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "f"'] & GA \arrow[d, "Gt"] \\ & GY \end{tikzcd}$$

in (*) as required.

So $\mathbb{A}$ is indeed weakly initial, and hence $(X \downarrow G)$ has a weakly initial set for all $X \in \mathcal{C}$. So finally, since $\mathcal{D}$ is locally small and complete, we can apply GAFT to see that G has a left adjoint. □

LECTURE 19 · 22/11/02

7 · Monads and comonads

7.1 · *Monads*

Suppose we have an adjunction $F \dashv G\colon \mathcal{D} \longrightarrow \mathcal{C}$. Write $T = GF\colon \mathcal{C} \longrightarrow \mathcal{C}$. We have natural transformations

$$\begin{aligned} \eta\colon 1_{\mathcal{C}} &\Rightarrow GF = T & \qquad \eta_X\colon X &\longrightarrow TX \\ G\varepsilon F\colon GFGF &\Rightarrow GF & & \\ \text{write as } \mu\colon T^2 &\Rightarrow T & \qquad \mu_X\colon T^2X &\longrightarrow TX \end{aligned}$$

We can think of $\eta\colon 1_{\mathcal{C}} \longrightarrow T$ as a "unit" and $\mu\colon T^2 \longrightarrow T$ as "multiplication".

PROPOSITION 7.1.1

Under the above conditions, the following diagrams commute:

1 Unit law:

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, "1"] \\ & T & \end{tikzcd} \quad \text{i.e. } \forall X \quad \begin{tikzcd} TX \arrow[r, "T\eta_X"] \arrow[dr, "1_{TX}"'] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "\eta_{TX}"'] \arrow[dl, "1_{TX}"] \\ & TX & \end{tikzcd} \quad \text{commutes.}$$

2 Associativity:

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \quad \text{i.e. } \forall X \quad \begin{tikzcd} T^3X \arrow[r, "\mu_{TX}"] \arrow[d, "T\mu_X"'] & T^2X \arrow[d, "\mu_X"] \\ T^2X \arrow[r, "\mu_X"'] & TX \end{tikzcd} \text{ commutes.}$$

PROOF

1

$$\begin{tikzcd} GFX \arrow[r, "GF\eta_X"] \arrow[dr, "1_{GFX}"'] & GFGFX \arrow[d, "G\varepsilon_{FX}"] & GFX \arrow[l, "\eta_{GFX}"'] \arrow[dl, "1_{GFX}"] \\ & GFX & \end{tikzcd}$$

commutes, since the left hand triangle is G of the triangle identity, and the right hand triangle is the triangle identity of FX.

2

$$\begin{tikzcd} GFGFGFX \arrow[r, "G\varepsilon_{FGFX}"] \arrow[d, "GFG\varepsilon_{FX}"'] & GFGFX \arrow[d, "G\varepsilon_{FX}"] \\ GFGFX \arrow[r, "G\varepsilon_{FX}"'] & GFX \end{tikzcd}$$

commutes as it is G of the naturality square of ε. □

DEFINITION 7.1.2

A *monad* on a category $\mathcal{C}$ consists of a functor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$ and natural transformations

$$\eta\colon 1 \Rightarrow T \qquad \text{"unit"}$$
$$\mu\colon T^2 \Rightarrow T \qquad \text{"multiplication"}$$

satisfying the unit and associativity laws as above.

EXAMPLES 7.1.3

1

$$\begin{aligned} (\ )^* : \mathbf{Set} &\longrightarrow \mathbf{Set} \\ A &\longmapsto A^* \end{aligned}$$

Where $A^* = \{ \text{lists } (a_1, \dots, a_n) \mid n \geqslant 0, \text{each } a_i \in A \}$. Put

$$\begin{aligned} \eta_A : A &\longrightarrow TA = A^* \\ a &\longmapsto (a) \end{aligned}$$

and

$$\begin{aligned} \mu_A : A^{**} &\longrightarrow A \\ ((a_{11}, \dots, a_{1n_1}), \dots, (a_{k1}, \dots, a_{kn_k})) &\longmapsto (a_{11}, \dots, a_{1n_1}, \dots, a_{k1}, \dots, a_{kn_k}) \end{aligned}$$

Then $((\ )^*, \eta, \mu)$ is a monad on **Set** - the "free monoid monad".

2 The identity functor is a monad.

3 Let $(M, e, \cdot)$ be a monoid. Then we have

$$M \times \_ : \mathbf{Set} \longrightarrow \mathbf{Set},$$

which we can equip with a monad structure. So set

$$\begin{aligned} \eta_X \colon X &\longrightarrow M \times X \\ x &\longmapsto (e, x) \\ \mu_X \colon M \times (M \times X) &\longrightarrow M \times X \\ (m_1, (m_2, x)) &\longmapsto (m_1 m_2, x) \end{aligned}$$

Then the unit and associativity laws for the monad follow precisely from those for the monoid.

DEFINITION 7.1.4

Dually we have *comonads*, a functor $L \colon \mathcal{D} \longrightarrow \mathcal{D}$ with $1_{\mathcal{D}} \stackrel{\varepsilon}{\Leftarrow} L \stackrel{\delta}{\Rightarrow} L^2$ satisfying the dual of the monad axioms.

7.2 · *Algebras for a monad*

DEFINITION 7.2.1

Let (T, η, μ) be a monad for $\mathcal{C}$. An *algebra* for T consists of an object $A \in \mathcal{C}$ together with a morphism $TA \xrightarrow{\theta} A \in \mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\theta"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T\theta"'] & TA \arrow[d, "\theta"] \\ TA \arrow[r, "\theta"] & A. \end{tikzcd}$$

A *map of algebras* $(TA \xrightarrow{\theta} A) \longrightarrow (TB \xrightarrow{\varphi} B)$ is a morphism $A \xrightarrow{f} B$ such that

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\theta"'] & TB \arrow[d, "\varphi"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. T-algebras and their maps form a category which we denote by $\mathcal{C}^T$.

EXAMPLES 7.2.2

1 $T = (\,)^* \colon \mathbf{Set} \longrightarrow \mathbf{Set}$. A T-algebra is precisely a monoid. For an algebra is a set A and a function $A^* \xrightarrow{\theta} A$ giving multiplication:

$$\begin{aligned} (a_1, a_2, \ldots, a_n) &\longmapsto a_1 a_2 \ldots a_n \\ (\,) &\longmapsto e \end{aligned}$$

The monad axioms tell us that the multiplication on A must be associative.

2 $T = \mathrm{id}$. Then $\mathcal{C}^T \cong \mathcal{C}$.

3 $T = M \times \_$. T-algebras are sets with a monoid action: $M \times A \xrightarrow{\theta} A$.

7.3 · *Free algebras*

We can define a forgetful functor:

$$U\colon \mathcal{C}^T \longrightarrow \mathcal{C}$$
$$(TA \xrightarrow{\theta} A) \longmapsto A$$
$$A \xrightarrow{f} B \longmapsto f$$

We may ask two obvious questions: does U have a left adjoint; and does T arise naturally from an adjunction?

PROPOSITION 7.3.1

U has a left adjoint $F\colon \mathcal{C} \longrightarrow \mathcal{C}^T$.

PROOF

We construct F as follows:

- on objects, $FA = \left(\begin{array}{c} T^2A \\ \downarrow{\scriptstyle \mu_A} \\ TA \end{array}\right)$, the "free algebra on A";
- on morphisms, $F(A \xrightarrow{f} B) = \left(\begin{array}{c} T^2A \\ \downarrow{\scriptstyle \mu_A} \\ TA \end{array}\right) \xrightarrow{Tf} \left(\begin{array}{c} T^2B \\ \downarrow{\scriptstyle \mu_B} \\ TB \end{array}\right)$.

We need to check three things: that FA and Ff satisfy the axioms for an algebra and a map of algebras; that F is functorial; and that F is left adjoint to U. So:

1 FA is a T-algebra:

$$\begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[dr, "1_{TA}"'] & T^2A \arrow[d, "\mu_A"] \\ & TA \end{tikzcd} \qquad \begin{tikzcd} T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & T^2A \arrow[d, "\mu_A"] \\ T^2A \arrow[r, "\mu_A"'] & TA \end{tikzcd}$$

by unit law for T by associativity law for T.

And Ff is a map of algebras:

$$\begin{tikzcd} T^2A \arrow[r, "T^2f"] \arrow[d, "\mu_A"'] & T^2B \arrow[d, "\mu_B"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

by naturality of μ.

2 The functoriality of F follows from that of T.

3 We need to show that

$$\mathcal{C}^T\left(FA, \begin{array}{c} TB \\ \downarrow{\scriptstyle \theta} \\ B \end{array}\right) \cong \mathcal{C}(A, B)$$

naturally in A and B. We construct an isomorphism as follows:

a Given a map of algebras in the LHS

$$\begin{tikzcd} T^2A \arrow[r, "Tg"] \arrow[d, "\mu_A"'] & TB \arrow[d, "\theta"] \\ TA \arrow[r, "g"'] & B, \end{tikzcd}$$

we take $A \xrightarrow{\eta_A} TA \longrightarrow B \in \mathcal{C}(A, B)$. Naturality follows from that of η.

b Given a morphism $A \xrightarrow{f} B$ in the RHS, we construct an algebra map

$$\begin{tikzcd} T^2A \arrow[r, "T^2f"] \arrow[d, "\mu_A"'] & T^2B \arrow[r, "T\theta"] \arrow[d, "\mu_B"] & Tb \arrow[d, "\theta"] \\ TA \arrow[r, "Tf"'] & TB \arrow[r, "\theta"'] & B \end{tikzcd}$$

The left hand square commutes by naturality of μ; the right hand square commutes by the second T-algebra axiom. Hence the outside square commutes, i.e. it is a map of algebras.

c We show that these are mutually inverse:

- Starting with $f\colon A \longrightarrow B$ on the RHS, we get taken to $\theta \circ Tf$ on the left hand side, and thence to $\theta \circ Tf \circ \eta_A$. So we need to show $\theta \circ Tf \circ \eta_A = f$. We have:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[drr, "f"'] & TA \arrow[r, "Tf"] & TB \arrow[r, "\theta"] & B \\ & & B \arrow[u, "\eta_B"] \arrow[ur, "1_B"'] & \end{tikzcd}$$

  The left hand square commutes by naturality of η and the right hand triangle by the first T-algebra axiom. So we are done.

- Starting with $g\colon TA \longrightarrow B$ on the LHS, we go to $g \circ \eta_A$ and then to $\theta \circ T(g \circ \eta_A)$. This time we have:

$$\begin{tikzcd} TA \arrow[r, "T\eta_A"] \arrow[dr, "1_{TA}"'] & T^2A \arrow[r, "Tg"] \arrow[d, "\mu_A"] & TB \arrow[r, "\theta"] & B \\ & TA \arrow[urr, "g"'] & & \end{tikzcd}$$

  where the left hand triangle commutes by the unit law for T, and the right hand square commutes since g is a T-algebra map.

So we have our adjunction as required. □

PROPOSITION 7.3.2

The adjunction $F \dashv U$ gives rise to the monad (T, η, μ).

PROOF

Recall that the adjunction $(F, U, \eta', \varepsilon')$ gives rise to a monad $(UF, \eta', U\varepsilon' F)$. So we need to check that $(UF, \eta', U\varepsilon' F) = (T, \eta, \mu)$.

1 It is easy to see that $UF = T$.

2 Recall the adjunction $(\mathcal{C}^T(FA, TB \xrightarrow{\theta} B) \cong \mathcal{C}(A, B)$ takes g to $g \circ \eta_A$. So the unit η'_A is given by

$$1_{FA} \mapsto 1_{FA} \circ \eta_A = \eta_A$$

as required.

3 Recall

$$\mathcal{C}\left(A, U\begin{pmatrix} TB \\ \downarrow \theta \\ B \end{pmatrix}\right) \cong \mathcal{C}^T\begin{pmatrix} FA, & \begin{matrix} TB \\ \downarrow \theta \\ B \end{matrix} \end{pmatrix}$$

has $f \mapsto \theta \circ Tf$. So the counit ε'_X at $X = TB \xrightarrow{\theta} B$ is given by

$$1_{UX} \mapsto \theta \circ T1 = \theta$$

We need to show $U\varepsilon'_{FA} = \mu_A$. But $FA = \begin{pmatrix} T^2A \\ \downarrow \mu_A \\ TA \end{pmatrix}$ so $U\varepsilon'_{FA} = \mu_A$ as required.

□

8 · Monadicity

8.1 · Introduction

DEFINITION 8.1.1

Given a monad $T \colon \mathcal{C} \longrightarrow \mathcal{C}$, we define a category Adj T with

- objects $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ inducing T;
- morphisms
$$\begin{tikzcd} & \mathcal{D}_1 \arrow[dl, shift left, "G_1"] \arrow[dd, "D"] \\ \mathcal{C} \arrow[ur, shift left, "F_1"] \arrow[dr, shift left, "F_2"] & \\ & \mathcal{D}_2 \arrow[ul, shift left, "G_2"] \end{tikzcd}$$
such that $F_2 = DF_1$ and $G_1 = G_2D$.

It is possible to show that in fact $F^T \dashv U^T$ is a terminal object in Adj T; so given $F \dashv G$, we get a unique morphism K in Adj GF:

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, shift left, "G"] \arrow[dd, dotted, "!K"] \\ \mathcal{C} \arrow[ur, shift left, "F"] \arrow[dr, shift left, "F^T"] & \\ & \mathcal{C}^T \arrow[ul, shift left, "U^T"] \end{tikzcd}$$

8.2 · *Eilenberg-Moore Comparison Functor*

DEFINITION 8.2.1

Given an adjunction $F \dashv G \colon \mathcal{D} \longrightarrow \mathcal{C}$, the Eilenberg-Moore comparison function K is the unique morphism K in Adj GF:

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, shift left, "G"] \arrow[dd, dotted, "!K"] \\ \mathcal{C} \arrow[ur, shift left, "F"] \arrow[dr, shift left, "F^T"] & \\ & \mathcal{C}^T \arrow[ul, shift left, "U^T"] \end{tikzcd}$$

and is given by:

- on objects, $KY = \begin{tikzcd} GFGY \arrow[d, "G\varepsilon_Y"] \\ GY \end{tikzcd}$
- on morphisms, $K(Y \xrightarrow{f} Z) = \begin{tikzcd} GFGY \arrow[r, "GFGf"] \arrow[d, "G\varepsilon_Y"'] & GFGZ, \arrow[d, "G\varepsilon_Z"] \\ GY \arrow[r, "Gf"'] & GZ \end{tikzcd}$

We need to check that K is in fact well defined; i.e. that KY is an algebra and that Kf is a map of algebras. For KY we have

$$\begin{tikzcd} GY \arrow[r, "\eta_{GY}"] \arrow[dr, "1_{GY}"'] & TGY \arrow[d, "G\varepsilon_Y"] \\ & GY \end{tikzcd}$$

which commutes by the first triangle identity, and

$$\begin{tikzcd} T^2GY \arrow[r, "\mu_{GY}"] \arrow[d, "TG\varepsilon_Y"'] & TGY \arrow[d, "G\varepsilon_Y"] \\ TGY \arrow[r, "G\varepsilon_Y"'] & GY \end{tikzcd} = \begin{tikzcd} GFGFGY \arrow[r, "G\varepsilon_{FGY}"] \arrow[d, "GFG\varepsilon_Y"'] & GFGY \arrow[d, "G\varepsilon_Y"] \\ GFGY \arrow[r, "G\varepsilon_Y"'] & GY \end{tikzcd}$$

which commutes by naturality of ε. Similarly for Kf, we have

$$\begin{tikzcd} GFGY \arrow[r, "GFGf"] \arrow[d, "G\varepsilon_Y"'] & GFGZ, \arrow[d, "G\varepsilon_Z"] \\ GY \arrow[r, "Gf"'] & GZ \end{tikzcd}$$

commuting by the naturality of ε. And clearly K is functorial (since G is), and the following diagrams commute:

$$\begin{tikzcd} & \mathcal{D} \arrow[dd, "K"] \\ \mathcal{C} \arrow[ur, "F"] \arrow[dr, "F^T"'] & \\ & \mathcal{C}^T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathcal{D} \arrow[dd, "K"] \arrow[dl, "G"'] \\ \mathcal{C} & \\ & \mathcal{C}^T \arrow[ul, "U^T"] \end{tikzcd}$$

DEFINITION 8.2.2

An adjunction $F \dashv G$ is called *monadic* if the Eilenberg-Moore comparison functor is an equivalence of categories. A functor G is called monadic if it has a left adjoint F with $F \dashv G$ monadic. A category $\mathcal{D}$ with an understood forgetful functor $\mathcal{D} \xrightarrow[U]{} \mathcal{C}$ is called monadic over $\mathcal{C}$ if U is monadic.

EXAMPLES 8.2.3

1. **Gp** is monadic over **Set**;
2. **Vect** is monadic over **Set**;
3. **Cpct Haus** is monadic over **Set**;
4. **Top** is not monadic over **Set**;
5. **Poset** is not monadic over **Set**.

8.3 · *Monadicity theorems*

Suppose we have an adjunction $F \dashv G\colon \mathcal{D} \longrightarrow \mathcal{C}$ giving rise to a monad $T = GF$. Asking whether $F \dashv G$ is monadic is essentially asking when $\mathcal{D}$ "looks like" $\mathcal{C}^T$, and when G "looks like" U^T. So what do $\mathcal{C}^T$ and U^T actually look like?

FACTS

1. Every algebra is a coequaliser of free algebras. Intuitively we can see this from "ordinary" algebra, where every algebra isa quotient of a free algebra. So monadicity theorems are all about existence, preservation, reflection and creation of special kinds of coequaliser.
2. U^T creates 'U^T-special' coequalisers. In fact this property characterises monadicity. Hence we arrive at our first attempt at a monadicity theorem:

THEOREM

G is monadic iff G creates G-special coequalisers.

Look more closely at (1). We want $\mathcal{D}$ to be like $\mathcal{C}^T$. So certainly we would like every object in $\mathcal{D}$ to be a coequaliser of free objects, i.e. objects of the form FX. This says that "the objects we do have look like algebras", i.e. that K is full and faithful.

We also need to show that we "have all of them", i.e. that K is essentially surjective. So does K hit all of the coequalisers? That is, can we find something in $\mathcal{D}$ which goes to each coequaliser? Well, if $\mathcal{D}$ has all the "special coequalisers" and G preserves them, then we can lift along U^T, so seeing that K sends it to the right place. Hence we get

THEOREM

$F \dashv G$ is monadic iff $\mathcal{D}$ has and G preserves G-very-special coequalisers, and every object of $\mathcal{D}$ is a coequaliser of free ones.

Can we avoid mentioning free objects in $\mathcal{D}$? In fact, the coequaliser in question is $\underset{FG\varepsilon_Y}{\overset{\varepsilon_{FGY}}{\rightrightarrows}} \xrightarrow{\varepsilon_Y}$; and G of this is a coequaliser in $\mathcal{C}$, so it suffices to prove that G reflects these. So K is full and faithful iff G reflects G-very-special coequalisers. Hence

THEOREM

G is monadic iff $\mathcal{D}$ has and G preserves and reflects G-very-special-coequalisers.

8.4 · *Background on coequalisers*

DEFINITION 8.4.1

A *split coequaliser* is a fork $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{e} C$ (i.e. $ef = eg$) with a splitting

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left, "t"] \arrow[r, "e"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

such that $es = 1_C$, $ft = 1_B$ and $gt = se$.

PROPOSITION 8.4.2

A split coequaliser is a coequaliser.

PROOF

Suppose we have a fork $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} D$, say, so that $hf = hg$. We need to show that there exists a unique $C \xrightarrow{k} D$ such that

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] \arrow[dr, "h"'] & C \arrow[d, dotted, "!k"] \\ & & D \end{tikzcd}$$

commutes. Now consider $hs\colon C \longrightarrow D$. We have

$$\begin{aligned} hse &= hgt \\ &= hft \\ &= h \end{aligned}$$

so hs certainly makes the diagram commute. And suppose k is any other such; then

$$ke = h = hse \quad \Rightarrow \quad kes = hses \quad \Rightarrow \quad k = hs$$

so hs is the unique such. $\square$

DEFINITION 8.4.3

An *absolute coequaliser* is a coequaliser that is preserved as a coequaliser by any functor.

PROPOSITION 8.4.4

A split coequaliser is an absolute coequaliser.

PROOF

A split coequaliser is defined entirely by a commutative diagram. $\square$

PROPOSITION 8.4.5

For any T-algebra $\begin{array}{c} TA \\ \downarrow{\scriptstyle\theta} \\ A \end{array}$, the following is a split coequaliser:

$$T^2A \overset{\mu_A}{\underset{T\theta}{\rightrightarrows}} TA \xrightarrow{\theta} A$$

PROOF

We exhibit a splitting $\begin{tikzcd} T^2A & TA \arrow[l, bend left, "\eta_{TA}"] & A \arrow[l, bend left, "\eta_A"] \end{tikzcd}$. For:

1 $\theta\eta_A = 1_A$ by the unit axiom for T-algebras.
2 $\mu_A\eta_{TA} = 1_{TA}$ by the unit axiom for the monad T.
3 $T\theta \circ \eta_{TA} = \eta_A \circ \theta$ by the naturality of η. $\square$

DEFINITIONS 8.4.6

- An *absolute coequaliser pair* is a pair $\overset{f}{\underset{g}{\rightrightarrows}}$ that has an absolute coequaliser.
- A *G-absolute coequaliser pair* is a pair f, g such that $\overset{Gf}{\underset{Gg}{\rightrightarrows}}$ has an absolute coequaliser.
- A *split coequaliser pair* is a pair $\overset{f}{\underset{g}{\rightrightarrows}}$ that has a split coequaliser.
- A *G-split coequaliser pair* is a pair f, g such that $\overset{Gf}{\underset{Gg}{\rightrightarrows}}$ has a split coequaliser.

In our earlier terminology, a "G-special coequaliser" is a coequaliser of a G-absolute-coequaliser pair. and a "G-very-special coequaliser" is a coequaliser of a G-split-coequaliser pair.

PROPOSITION 8.4.7

$FGFGY \overset{\varepsilon_{FGY}}{\underset{FG\varepsilon_Y}{\rightrightarrows}} FGY$ is a G-split coequaliser pair.

PROOF

Recall $KY = \begin{array}{c} GFGY \\ \downarrow{\scriptstyle G\varepsilon_Y} \\ GY \end{array}$ is an algebra. Hence by previous result

$$GFGFGY \overset{G\varepsilon_{FGY}}{\underset{GFG\varepsilon_Y}{\rightrightarrows}} GFGY \xrightarrow{G\varepsilon_Y} GY$$

is a split coequaliser. $\square$

8.5 · *Beck's Monadicity Theorem*

THEOREM 8.5.1

Let $F \dashv G \colon \mathcal{D} \longrightarrow \mathcal{C}$. Then the following are equivalent:

1 The adjunction is monadic;
2 G creates coequalisers for all G-absolute-coequaliser pairs;
3 $\mathcal{D}$ has coequalisers of all G-split coequaliser pairs, and G preserves and reflects them.

To prove this, we shall first prove a series of propositions.

PROPOSITION 8.5.2

$U^T \colon \mathcal{C}^T \longrightarrow \mathcal{C}$ creates coequalisers for all U^T-absolute-coequaliser pairs.

PROOF

A U^T-absolute-coequaliser pair is a pair of morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$ such that

$$\begin{tikzcd} TA \arrow[r, shift left, "Tf"] \arrow[r, shift right, "Tg"'] \arrow[d, "\theta"'] & TB \arrow[d, "\varphi"] \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

"serially commutes", and such that $A \underset{g}{\overset{f}{\rightrightarrows}} B$ has an absolute coequaliser $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{e} C$ in $\mathcal{C}$.

We aim to show that there is a unique lift to a fork

$$\begin{tikzcd} TA \arrow[r, shift left, "Tf"] \arrow[r, shift right, "Tg"'] \arrow[d, "\theta"'] & TB \arrow[r, "Te"] \arrow[d, "\varphi"] & TC \arrow[d, "\psi"] \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] & C \end{tikzcd}$$

in $\mathcal{C}^T$, and that it is a coequaliser in $\mathcal{C}^T$.

1 Induce unique ψ by the universal property of coequaliser; the bottom fork is an absolute coequaliser, hence preserved by T; so the top fork is also a coequaliser. Now,

$$e \circ \varphi \circ Tf = e \circ f \circ \theta = e \circ g \circ \theta = e \circ \varphi \circ Tg$$

so this induces a unique ψ making the right hand square commute.

2 We show that $TC \xrightarrow{\psi} C$ is an algebra. For the first axiom, consider the diagram:

$$\begin{tikzcd} B \arrow[rr, "e"] \arrow[dr, "\eta_B"] \arrow[dd, "1_B"'] & & C \arrow[dr, "\eta_C"] \arrow[dd, "1_C"' near start] & \\ & TB \arrow[rr, "Te"] \arrow[dl, "\varphi"] & & TC \arrow[dl, "\psi"] \\ B \arrow[rr, "e"'] & & C & \end{tikzcd}$$

We need to show the right hand triangle commutes. But everything else commutes, and e is epic (since a coequaliser). Hence the right hand triangle commutes. Similarly, for the second axiom, consider:

$$\begin{tikzcd}
T^2B \arrow[rr, "T^2e"] \arrow[dr, "\eta_B"] \arrow[dd, "T\varphi"'] & & T^2C \arrow[dr, "\mu_C"] \arrow[dd, "T\psi"] & \\
& TB \arrow[rr, "Te"] \arrow[dd, "\varphi"] & & TC \arrow[dd, "\psi"] \\
TB \arrow[rr, "Te"] \arrow[dr, "\varphi"'] & & TC \arrow[dr, "\psi"] & \\
& B \arrow[rr, "e"] & & C
\end{tikzcd}$$

We need to show the right hand face commutes. But everything else commutes and T^2e is epic (since a coequaliser); hence the right hand square does commute.

3 It remains to check that the given fork is a coequaliser in $\mathcal{C}^T$. Consider:

$$\begin{tikzcd}
TA \arrow[r, shift left, "Tf"] \arrow[r, shift right, "Tg"'] \arrow[dd, "\theta"'] & TB \arrow[rr, "Te"] \arrow[dr, "Th"'] \arrow[dd, "\varphi"] & & TC \arrow[dl, dotted, "T\overline{h}"] \arrow[dd, "\psi"] \\
& & TY \arrow[dd, "\alpha"] & \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[rr, "e"] \arrow[dr, "h"'] & & C \arrow[dl, dotted, "!\overline{h}"] \\
& & Y &
\end{tikzcd}$$

where we induce the unique $\overline{h}$ by the bottom coequaliser. Then since Te is epic, the right hand square commutes, exhibiting $\overline{h}$ as a unique factorisation in $\mathcal{C}^T$ as required. $\square$

PROPOSITION 8.5.3

For any algebra $TA \xrightarrow{\theta} A$, the following diagram is a coequaliser in $\mathcal{C}^T$:

$$\begin{tikzcd}
T^3A \arrow[r, shift left, "T\mu_A"] \arrow[r, shift right, "T^2\theta"'] \arrow[d, "\mu_{TA}"'] & T^2A \arrow[r, "T\theta"] \arrow[d, "\mu_A"] & TA \arrow[d, "\theta"] \\
T^2A \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\theta"'] & TA \arrow[r, "\theta"'] & A
\end{tikzcd}$$

PROOF

Observe that this diagram serially commutes, i.e. it is a fork. Also note that U^T of it is an absolute coequaliser (by Prop 8.4.5). Since U^T creates and in particular reflects coequalisers for U^T-absolute coequaliser pairs, this fork must itself be a coequaliser. $\square$

PROPOSITION 8.5.4

K is full and faithful iff the following diagram is a coequaliser for all $A \in \mathcal{D}$:

$$FGFGA \underset{FG\varepsilon_A}{\overset{\varepsilon_{FGA}}{\rightrightarrows}} FGA \xrightarrow{\varepsilon_A} A$$

PROOF

The right hand side says: given any $m\colon FGA \longrightarrow B$ such that $m \circ \varepsilon_{FGA} = m \circ FG\varepsilon_A$, there exists a unique $f\colon A \longrightarrow B$ such that $f \circ \varepsilon_A = m$. The left hand side says:

$$K\colon \mathcal{D}(A, B) \longrightarrow \mathcal{C}^T(KA, KB)$$
$$f \longmapsto Gf$$

is a bijection for all $A, B \in \mathcal{D}$ (recall $Kf = Gf$). That is, given any $h\colon KA \longrightarrow KB$, there is a unique $f\colon A \longrightarrow B$ such that $h = Gf$. But:

CLAIM

A map $h\colon KA \longrightarrow KB$ is precisely a map $GA \xrightarrow{h} GB$ such that $\overline{h} \circ \varepsilon_{FGA} = \overline{h} \circ FG\varepsilon_A$.

PROOF

Such an h makes

$$\begin{tikzcd} GFGA \arrow[r, "GFh"] \arrow[d, "G\varepsilon_A"'] & GFGB \arrow[d, "G\varepsilon_B"] \\ GA \arrow[r, "h"] & GB \end{tikzcd}$$

commute; i.e. $h \circ G\varepsilon_A = G\varepsilon_B \circ GFH$. Now:

$$\frac{GFGA \xrightarrow{G\varepsilon_A} GA \xrightarrow{h} GB}{FGFGA \xrightarrow{FG\varepsilon_A} FGA \xrightarrow{\overline{h}} GB}$$

along the leftish leg, and

$$\frac{GFGA \xrightarrow{1_{GFGA}} GFGA \xrightarrow{GFh} GFGB \xrightarrow{G\varepsilon_B} GB}{FGFGA \xrightarrow{\varepsilon_{FGA}} FGA \xrightarrow{Fh} FGB \xrightarrow{\varepsilon_B} B}$$

along the rightish one; but $\varepsilon_B \circ Fh = \overline{h}$, so the condition becomes $\overline{h} \circ \varepsilon_{FGA} = \overline{h} \circ FG\varepsilon_A$. □

But now, under adjunction, $h\colon GA \longrightarrow GB$ becomes $\overline{h}\colon FGA \longrightarrow B$, and $Gf\colon GA \longrightarrow GB$ becomes $f \circ \varepsilon_A\colon FGA \longrightarrow B$. Hence, the left hand side statement becomes: given any $\overline{h}\colon FGA \longrightarrow B$ such that $\overline{h} \circ \varepsilon_{FGA} = \overline{h} \circ FG\varepsilon_A$, there exists unique $f\colon A \longrightarrow B$ such that $\overline{h} = f \circ \varepsilon_A$, which is precisely the right hand side statement. □

PROPOSITION 8.5.5

K is full and faithful if G reflects coequalisers for all G-split coequaliser pairs.

PROOF

G of $FGFGA \underset{FG\varepsilon_A}{\overset{\varepsilon_{FGA}}{\rightrightarrows}} FGA$ is a split coequaliser by 8.4.7. So if G reflects such coequalisers, then this fork is a coequaliser. And hence K is full and faithful by the previous result. □

PROPOSITION 8.5.6

If $\mathcal{D}$ has and G preserves coequalisers for all G-split coequaliser pairs, then K is essentially surjective.

PROOF

Given any algebra $TA \xrightarrow{\theta} A$, we seek $Y \in \mathcal{D}$ such that $KY \cong TA \xrightarrow{\theta} A$ in $\mathcal{C}^T$. Recall that

$$\begin{tikzcd} T^3A \arrow[r, shift left, "T\mu_A"] \arrow[r, shift right, "T^2\theta"'] \arrow[d, "\mu_{TA}"'] & T^2A \arrow[r, "T\theta"] \arrow[d, "\mu_A"] & TA \arrow[d, "\theta"] \\ T^2A \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\theta"'] & TA \arrow[r, "\theta"'] & A \end{tikzcd} \qquad (1)$$

is a coequaliser in $\mathcal{C}^T$, and that the left hand square is a U^T-split coequaliser pair (since the bottom is a split coequaliser pair by 8.4.5).

Also by 8.4.5, $FGFA \overset{\varepsilon_{FA}}{\underset{F\theta}{\rightrightarrows}} FA$ is a G-split coequaliser pair, and K of it is the pair in (1) (since $K \circ U^T = G$).

So it has a coequaliser in $\mathcal{D}$,

$$FGFA \overset{\varepsilon_{FA}}{\underset{F\theta}{\rightrightarrows}} FA \xrightarrow{h} Y \qquad (2)$$

say. We show that K of this coequaliser is a coequaliser of the same parallel pair we started with. Recall the following diagram commutes:

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "K"] \arrow[dr, "G"'] & & \mathcal{C}^T \arrow[dl, "U^T"] \\ & \mathcal{C} & \end{tikzcd}$$

G preserves coequalisers of G-split coequaliser pairs; so G of (2) is a coequaliser in $\mathcal{C}$. K of the pair is a U^T-split-coequaliser pair; U^T creates coequalisers for such. So K of (2) is a coequaliser. Hence it must be isomorphic to (1); i.e. $KY \cong (TA \xrightarrow[\theta]{} A)$. □

We are now in a position to prove Beck's Monadicity Theorem.

PROOF (OF 8.5.1)

1 $\Rightarrow$ 2: Since U^T creates coequalisers for U^T-absolute coequaliser pairs, and K is an equivalence of categories, so the same holds for G.

2 $\Rightarrow$ 3: Immediate from definitions; a split coequaliser is an absolute coequaliser, and "creates" implies "reflects"; so G preserves and reflects split coequalisers.
Since G creates split coequalisers, $\mathcal{D}$ has them. And this was of getting coequalisers in $\mathcal{D}$ does give all the coequalisers we want, so by construction all these are taken to coequalisers in $\mathcal{C}$.

3 $\Rightarrow$ 1: by Prop 8.5.5 and 8.5.6.

□

9 · Bicategories

9.1 · Definitions

DEFINITION 9.1.1

A *category* $\mathcal{C}$ is given by:

- DATA:
  - a collection ob $\mathcal{C}$ of objects;
  - for each pair of objects, a collection of morphisms $\mathcal{C}(A, B)$;
  - for each $A, B, C \in \text{ob}\,\mathcal{C}$, a function

$$\begin{aligned} c_{ABC} \colon \mathcal{C}(B, C) \times \mathcal{C}(A, B) &\longrightarrow \mathcal{C}(A, C) \\ (g, f) &\longmapsto g \circ f; \end{aligned}$$

  - for each $A \in \mathcal{C}$, a function

$$\begin{aligned} i_A \colon \mathcal{C}(A, A) & \\ * &\longmapsto \text{id}_A\,. \end{aligned}$$

- AXIOMS:
  - associativity — $(hg)f = h(gf)$;
  - unit — $f \circ 1 = f = 1 \circ f$.

DEFINITION 9.1.2

A *bicategory* $\mathcal{B}$ is given by

- DATA:
  - a collection ob $\mathcal{B}$ of 0-cells;
  - for each pair A, B of 0-cells, a category $\mathcal{B}(A, B)$, with
    - objects being 1-cells $A \longrightarrow B$;
    - morphisms being 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$;
    - composition $\begin{tikzcd}[row sep=small] A \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & B \end{tikzcd}$, $\beta \circ \alpha$.
  - composition: for each $A, B, C \in \mathcal{B}$, a functor

$$\begin{array}{rccc} c_{ABC} \colon & \mathcal{B}(B, C) \times \mathcal{B}(A, B) & \longrightarrow & \mathcal{B}(A, C) \\ & (g\,,\,f) & \longmapsto & gf \end{array}$$

$$\left(\begin{tikzcd} B \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} \,,\, \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \right) \longmapsto \begin{tikzcd} A \arrow[r, bend left, "gf"] \arrow[r, bend right, "g'f'"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & C \end{tikzcd}$$

– identities: for each $A \in \mathcal{B}$, a functor

$$\begin{aligned} I_A \colon 1 &\longrightarrow \mathcal{B}(A,A) \\ * &\longmapsto A \underset{I_A}{\longrightarrow} A \end{aligned}$$

– associativity: for all composable $f, g, h \in \mathcal{B}$, invertible 2-cells

$$\mathfrak{a}_{fgh} \colon (hg)f \xrightarrow{\sim} h(gf)$$

natural in f, g and h.

– unit: for all $f \in \mathcal{B}(A,B)$:

$$\begin{aligned} \mathfrak{r}_f \colon f \circ I_A &\xrightarrow{\sim} f \\ \mathfrak{l}_f \colon I_B \circ f &\xrightarrow{\sim} f \end{aligned}$$

natural in f.

- AXIOMS:
  – the associativity pentagon commutes:

$$\begin{tikzcd} & ((kh)g)f \arrow[rr, "\mathfrak{a}*1"] \arrow[dl, "\mathfrak{a}"'] & & (k(hg))f \arrow[dr, "\mathfrak{a}"] & \\ (kh)(gf) \arrow[drr, "\mathfrak{a}"'] & & & & k((hg)f) \arrow[dll, "1*\mathfrak{a}"] \\ & & k(h(gf)) & & \end{tikzcd}$$

  – the unit triangle commutes:

$$\begin{tikzcd} (gI)f \arrow[rr, "\mathfrak{a}"] \arrow[dr, "\mathfrak{r}*1"'] & & g(If) \arrow[dl, "1*\mathfrak{l}"] \\ & gf & \end{tikzcd}$$

EXAMPLES 9.1.3

1 If $\mathfrak{a}$, $\mathfrak{r}$ and $\mathfrak{l}$ are identities, we have a strict 2-category; for example **Cat**.
2 A bicategory with one object is called a *monoidal category*.
3 **Set** has the structure of a monoidal category.

1-object bicategory	$\leftrightarrow$	monoidal category
1-cells	$\leftrightarrow$	objects
2-cells	$\leftrightarrow$	morphisms
composition of 1-cells	$\leftrightarrow$	"tensor product" of objects $A \otimes B$

In **Set** we take $A \otimes B = A \times B$ the usual Cartesian product. Then $\mathfrak{a} \colon A \times (B \times C) \xrightarrow{\sim} A \times (B \times C)$; and we take I to be an object such that $A \times I \cong A \cong I \times A$; i.e. any one-object set.
4 There is a bicategory of rings, bimodules and bimodule homomorphisms.
5 Any category can be regarded as a bicategory with trivial 2-cells.

9.2 · *Slightly higher-dimensional categories*

DEFINITION 9.2.1

A *monoidal category* is a category $\mathcal{C}$ equipped with

- a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$;
- an object $I \in \operatorname{ob} \mathcal{C}$

together with natural isomorphisms

$$\begin{aligned} \mathfrak{a}_{ABC} \colon (A \otimes B) \otimes C &\xrightarrow{\sim} A \otimes (B \otimes C) \\ \mathfrak{l}_A \colon I \otimes A &\xrightarrow{\sim} A \\ \mathfrak{r}_A \colon A \otimes I &\xrightarrow{\sim} A \end{aligned}$$

such that the following diagrams commute:

$$\begin{tikzcd} & ((A \otimes B) \otimes C) \otimes D \arrow[r, "\mathfrak{a} \otimes 1"] \arrow[dl, "\mathfrak{a}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[dr, "\mathfrak{a}"] & \\ (A \otimes B) \otimes (C \otimes D) \arrow[drr, "\mathfrak{a}"'] & & & A \otimes ((B \otimes C) \otimes D) \arrow[dl, "1 \otimes \mathfrak{a}"] \\ & & A \otimes (B \otimes (C \otimes D)) & \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes I) \otimes B \arrow[rr, "\mathfrak{a}"] \arrow[dr, "\mathfrak{r} \otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1 \otimes \mathfrak{l}"] \\ & A \otimes B & \end{tikzcd}$$

EXAMPLE 9.2.2

Given any category $\mathcal{C}$ we can form a monoidal category from it:

- objects are finite lists $(x_1, \dots, x_n)$ of objects of $\mathcal{C}$;
- morphisms $(x_1, \dots, x_m) \xrightarrow{(f_1, \dots, f_m)} (y_1, \dots, y_m)$ with $f_i \colon x_i \longrightarrow y_i$.

I is the empty list, and $\otimes$ is concatenation of lists. This is known as the "free strict monoidal category on $\mathcal{C}$".

We can draw morphisms as

$$\begin{tikzcd}[row sep=small, column sep=tiny] x_1, \arrow[d, "f_1"] & x_2, \arrow[d, "f_2"] & \dots, & x_m \arrow[d, "f_m"] \\ y_1, & y_2, & \dots, & y_m \end{tikzcd}$$

We have seen other examples of monoidal categories; for instance, **Set** with $A \otimes B = A \times B$. However, in this case we could have equally well chosen to use $B \times A$, since we have $A \times B \cong B \times A$ — a *symmetry*

DEFINITION 9.2.3

A *symmetry* for a monoidal category $(\mathcal{C}, \otimes, I, \mathfrak{a}, \mathfrak{r}, \mathfrak{l})$ is given by isomorphisms

$$\gamma_{AB} \colon A \otimes B \xrightarrow{\sim} B \otimes A$$

natural in A and B such that the following diagrams commute:

$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[r, "\mathfrak{a}"] \arrow[dl, "\gamma\otimes 1"'] & A \otimes (B \otimes C) \arrow[dr, "\gamma"] & \\
(B \otimes A) \otimes C \arrow[dr, "\mathfrak{a}"'] & & & (B \otimes C) \otimes A \arrow[dl, "\mathfrak{a}"] \\
 & B \otimes (A \otimes C) \arrow[r, "1\otimes\gamma"] & B \otimes (C \otimes A) &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\gamma"] \arrow[dr, "\mathfrak{r}"'] & & I \otimes A \arrow[dl, "\mathfrak{l}"] \\
 & A &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes B \arrow[r, "\gamma"] \arrow[dr, "1"'] & B \otimes A \arrow[d, "\gamma"] \\
 & A \otimes B
\end{tikzcd}$$

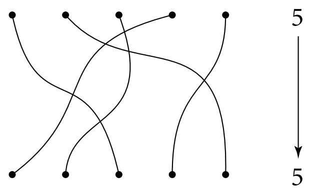
We call such a category a *symmetric monoidal category*.

EXAMPLE 9.2.4

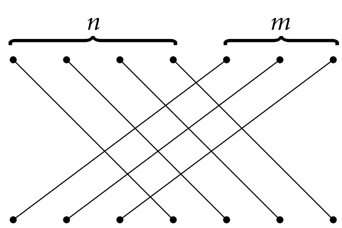
Let $\mathcal{C}$ be the category with objects the natural numbers and morphisms given by

$$\mathcal{C}(n, m) = \begin{cases} S_n & n = m \\ \varnothing & n \neq m \end{cases}$$

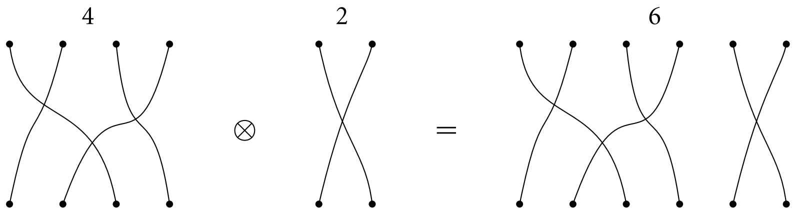
So we can draw morphisms as



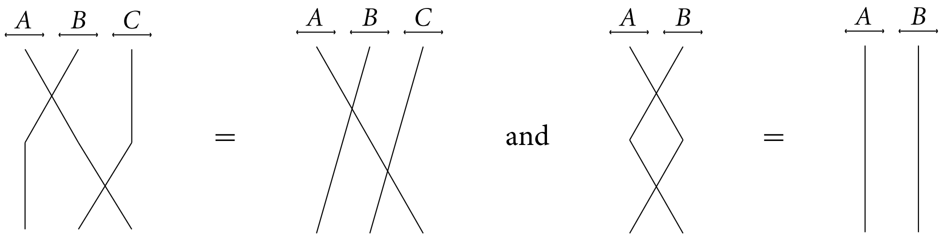
and we can compose them. Now, we can make $\mathcal{C}$ into a symmetric monodial category by defining $\otimes$ on objects to be addition (a strictly associative map!), I to be 0, and γ_{nm} given by



We define $\otimes$ on morphisms to be juxtaposition of permutations; for example



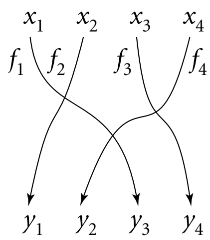
And our axioms say



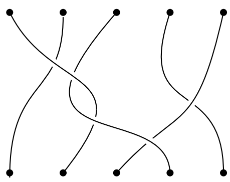
which is 'pictorially obvious'. In fact, any two morphisms that are 'pictorially the same' are the same.

EXAMPLE 9.2.5

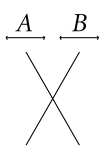
Just as for monoidal categories, we can form the "free strictly associative symmetrical monoidal category" on a category $\mathcal{C}$. The objects are finite lists, and the morphisms are as in the previous example, but labelled by morphisms of $\mathcal{C}$; for example



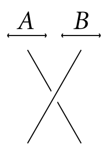
Note that we do not distinguish over- and under-crossings. But we could; so we would have diagrams that looked like



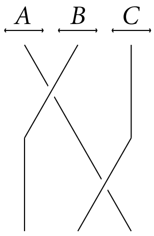
That is, instead of our symmetry being



it is



Note that one of the axioms for a symmetry does not now hold; we still have



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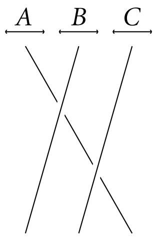
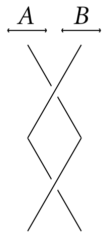
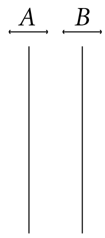


but



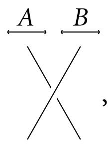
$\neq$



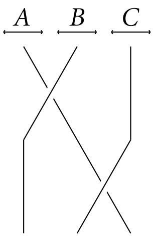
DEFINITION 9.2.6

A *braided monoidal category* is a monoidal category equipped with a *braiding*; that is, isomorphisms

$$\mathfrak{c}_{AB} \colon A \otimes B \longrightarrow B \otimes A$$

natural in A and B, and denoted by 

, such that



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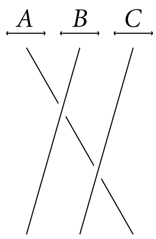
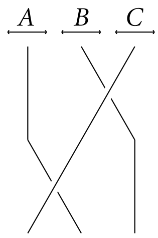


and



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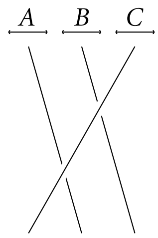
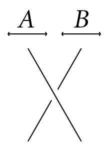


Note that we have another braiding

$$\mathfrak{c}'_{AB} = \mathfrak{c}_{BA}^{-1} \qquad \text{i.e.}$$



but in general $\mathfrak{c} \neq \mathfrak{c}'$; if the two are equal, then we in fact have a symmetry.

Note that in the symmetric case we did not have to specify both of the above axioms, as one was the inverse of the other.

REMARK

As before, we can form a "free braided monoidal category" on $\mathcal{C}$ by labelling strands. Then to check that diagrams commute we check each strand and check that the underlying braids are the same.