Gros and Petit Toposes

Nick Duncan

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

Introduction

- Gros and Petit Toposes
- Zariski Topos
- Connectedness in Geometry

Gros and Petit Toposes: Intuitive Definition

- Gros Topos: A topos whose objects have a geometric structure
- Examples: Simplicial sets, $\mathrm{Set}^{\triangle^{\mathrm{op}}}$. There is a geometric realization functor, which provides every object with a topological object.
  TOP($\mathscr{T}$), a topos on a full subcategory $\mathscr{T}$ of topological spaces, closed under subobjects and finite limits. Each object is colimit of representables, and each representable is a topological space, so every object is a "formal colimit" of topological spaces.
- Petit Topos: A topos constructed from a single geometric object.
- Examples: Sheaves on a topological space X, $\mathrm{Sh}(X)$.
  Sets with a group action: G-Set.

Toposes and Geometry

- Toposes allow us to perform a large number of constructions.
- Important constructions: Products, Pullbacks, Equalizers, Disjoint Unions, Quotients, Function Spaces.
- To use topos theory in geometry start with a small category of basic geometric objects.
- We can consider presheaves over this category, but it will give us the wrong colimits.
- In order to get the right colimits we need an appropriate coverage on our basic geometric objects.
- Examples of basic geometric objects: Manifolds, small categories of topological spaces, analytic spaces, semialgebraic sets (solutions to polynomial inequalities).

Gros and Petit Toposes: Rigorous Definition

Definition

A Gros topos consists of a topos $\mathscr{G}$, and for every object F of $\mathscr{G}$, a topos $\mathscr{P}(F)$, called the petit topos of F, along with a local geometric morphism $p_F : \mathscr{G}/F \to \mathscr{P}(F)$, such that for any morphism $g : F \to H$, there is a geometric morphism $\mathscr{P}(g) : \mathscr{P}(F) \to \mathscr{P}(H)$ which satisfies a coherence condition for composition of morphisms.

- Local geometric morphism: The direct image p_* has a right adjoint $p^!$, which is full and faithful.
- A local geometric morphism induces a topos inclusion, given by $q^* = p_*$, $q_* = p^!$.
- This induces a "homotopy equivalence" of toposes between $\mathscr{G}/F$ and $\mathscr{P}(F)$. Hence they have the same cohomology, which allows us to calculate the cohomology using the smaller petit topos.

The Petit Topos $\mathscr{P}(1)$

- The petit topos associated to the terminal object is an important topos.
- We call it the topos of points of the gros topos $\mathscr{G}$.
- Objects of the form $p^*(A)$ in $\mathscr{G}$ are called discrete, and objects of the form $p^!(A)$ are called codiscrete.
- Discrete and codiscrete objects both form full subcategories.

Solutions of Polynomial Equations

- The beginning of Algebraic Geometry is the solution of polynomial equations:

$$
\begin{aligned}
p_1(X_1, X_2, \ldots, X_n) &= 0 \\
p_2(X_1, X_2, \ldots, X_n) &= 0 \\
&\ldots \\
p_m(X_1, X_2, \ldots, X_n) &= 0
\end{aligned}
$$

- The solutions are interpreted in affine space of some base ring, usually an algebraically closed field k. Affine space is the set $\mathbb{A}^n = k \times \cdots \times k$.
- The solutions of these functions give rise to a geometric object in affine space, called an algebraic set.
- The basic geometric objects studied in algebraic geometry are algebraic sets.

Topological Structure of Affine Space

- Affine space $\mathbb{A}^n$ has a topological structure. Closed sets are the algebraic sets.
- Each algebraic set has an induced topological structure.
- In order to put a coverage on this category we need open objects. Only some algebraic sets are homeomorphic to an open subset, the ones we consider are the called the distinguished open sets.
- Distinguished open sets are the solutions to the following polynomial: $(fY - 1)$, where f is any polynomial.
- Example: $(XY - 1)$ in $\mathbb{A}^2$ is homeomorphic to the complement of the closed point $(X - 0)$ in $\mathbb{A}^1$.

Algebraic Sets vs k-Algebras

Correspondence between reduced k-algebras and algebraic sets

Hilbert's Nullstellensatz gives the following correspondence:

$$
\begin{aligned}
V \subseteq \mathbb{A}^n &\quad \rightsquigarrow \quad k[X_1, \ldots, X_n]/(f \mid f(x) = 0 \; \forall x \in V) \\
k[X_1, \ldots, X_n]/I &\quad \rightsquigarrow \quad \{x \mid f(x) = 0 \; \forall f \in I\}
\end{aligned}
$$

- Using this correspondence we extend our basic geometric objects to be the category of (finitely presented) k-algebras.
- We include non-reduced k-algebras, as they give us some objects which have interesting geometry which cannot be expressed as an algebraic set.
- One example is the algebra of dual numbers, $k[\varepsilon] = k[X]/(X^2)$. This object allows us to easily define tangent vectors.

The Zariski Topology

- We need to extend our topology on algebraic sets. The points in the topology of an algebraic set correspond to maximal ideals of the corresponding k-algebra, and the closed sets can be constructed from radical ideals. $V(I) = \{\mathfrak{m} | I \subseteq \mathfrak{m}\}$
- In order to extend this to an arbitrary k-algebra A we take prime ideals of A as our points. Radical ideals correspond to reduced k-algebras, so we take closed sets constructed from arbitrary ideals. For a given ideal I, $V(I) = \{\mathfrak{p} | I \subseteq \mathfrak{p}\}$
- This topology is called the Zariski topology, and for a k-algebra A it is denoted Spec (A).
- The distinguished opens form a basis for the Zariski topology. For a given k-algebra A, these are the k-algebras $A_f = A[Y]/(fY - 1)$, for any $f \in A$.

The Zariski Topos

- In order to construct limits, colimits and function spaces, we embed the category of algebraic sets into a topos.
- We construct a co-site for this topos. Let k be a field, not necessarily algebraically closed.

Co-site for k-Zar

Objects:	finitely presented k-algebras.
Morphisms:	k-algebra morphisms.
Co-coverage:	induced by families $(A \to A[a_i^{-1}] \| i = 1, \ldots, n)$, where the ideal (a_i) equals A.

Note $A[a_i^{-1}] = A[X]/(a_i X - 1)$, which correspond to distinguished opens. These form a basis for the Zariski topology.

Objects of the Zariski Topos

- The Zariski topos contains many objects, some geometrical, others not visibly so.
- Any algebraic set is in the topos, via the Yoneda embedding.
- Non-reduced algebras, like the dual numbers, $k[\varepsilon] = k[X]/(X^2)$, which geometrically looks like a "disembodied tangent vector".
- Schemes live in this topos, which are sheaves which are covered by representables. An example is projective space. $\mathbb{P}^n(A) = \{$ rank 1 direct factors of $A^{n+1}\}$.
- Non-separated schemes, like the affine line with a point doubled.
- Finally, non scheme-like objects also live in this topos, which represent pathological geometric conditions, like a line with two tangent directions at a point.

The Gros Structure of the Zariski Topos

- We consider the Zariski topos as a Gros topos - its objects have geometrical structure.
- We give a Petit Topos structure to each object in k-Zar.
- For any k-algebra, we associate the topos $\text{Sh}(\text{Spec } A)$.
- We can represent any sheaf F in k-Zar as a colimit of representables. We associate a topological space $|F|$, as the colimit of Spec A calculated in the category of topological spaces.
- The petit topos of points is just the topos Set, as Spec of any field is a point.

Examples of Objects in Gros and Petit Zariski Toposes

- Consider the presheaf $\mathscr{O} : A \mapsto$ underlying set of A.
- This is represented by the affine line $k[X]$.
- For any object F in k-Zar, we get an object $F^*\mathscr{O}$ in $k\text{-Zar}/F$.
- Restricting this object to the Petit topos gives us a sheaf $\mathscr{O}_F$ in $\mathrm{Sh}(|F|)$.
- If F is a k-algebra, or more generally a scheme X, then $\mathscr{O}_F$ is the standard structure sheaf of X.
- The pair $(|X|, \mathscr{O}_X)$ is a locally ringed space, and has all the structural components of a scheme. Hence the scheme-theoretic definitions in algebraic geometry appear in the petit toposes.

Points in Algebraic Geometry

- The concept of a point has always been difficult in algebraic geometry.
- We have already changed our notion of point by going from the topology of algebraic sets to the Zariski topology.
- However, even this may not be enough in certain cases. Example: $\mathbb{R}$ and $\mathbb{R}[X]/(X^2+1) = \mathbb{C}$ are both fields, so have the same Spec.
- We will use a Petit topos to measure points.
- Example: $\mathbb{R}$-Zar has two types of points: $\mathbb{R}$-points, which correspond to global points of the gros topos, and $\mathbb{C}$-points.
- In general, the petit topos $\mathscr{P}(1) = \mathrm{Set}$ is not enough to measure points in k-Zar, when k is not algebraically closed.

Connected Components in a Gros Topos

- In topology, we say that a space X is connected if every continuous function from X to a discrete space is constant.
- For objects in a gros topos we can use a similar definition.
- The discrete objects are those of the form $p^*(S)$, where p is the geometric morphism $p : \mathscr{G} \to \mathscr{P}$.
- We measure connectedness via the petit topos $\mathscr{P}$.

$$\mathscr{P}(\pi(F), S) \cong \mathscr{G}(F, p^*(S))$$

- Hence we measure connectedness via a left adjoint, $\pi = p_!$, of p^*.

Locally Connected Geometric Morphisms

Locally Connected

A geometric morphism $p : \mathscr{G} \to \mathscr{P}$ is locally connected when the inverse image p^* has a ($\mathscr{P}$-indexed) left adjoint $p_!$.

- An object is called connected if $p_!(F) = 1$.
- Any object in $\mathscr{G}$ is a ($\mathscr{P}$-indexed) coproduct of connected objects.
- A locally connected geometric morphism is a connected geometric morphism if $p_!(1) = 1$.
- It is a reasonable geometric assumption for a point to be connected.

Connected Components and Finite Products

- One final property we want for our connected component functor is preservation of finite products. $p_!(F \times G) \cong p_!(F) \times p_!(G)$
- It is a geometrically reasonable idea that the connected components of a product is a product of the individual connected components.
- It is also important for group structures and other algebra structures. The connected components of an internal group should also form an internal group.
- However, we do not want $p_!$ to preserve equalizers, or subobjects. Example: The affine line $k[X]$ has the two points $(X - 0)$ and $(X - 1)$. The subobject consisting of these two points is the algebra $k \times k$, which is not connected. However, the line is connected.

Finding Suitable Petit Topos Structures

- We will look for suitable objects of connected components within our site.
- In the Zariski topos, Spec (A) of a k-algebra A is connected iff A has no non-trivial idempotents. We could use Spec to measure connectedness, however this does not preserve products.
  In $\mathbb{R}$-Zar, Spec $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \{id, conj\}$, where as
  Spec $(\mathbb{C})\times$ Spec $(\mathbb{C}) = \{*\} \times \{*\} = \{*\}$.
- Instead we use separable k-algebras to measure connected components.
- These are finitely presented k-algebras A such that $k_s \otimes_k A \cong k_s \times \cdots \times k_s$, where k_s is the separable closure of the field k. We could also use the algebraic closure $\bar{k}$.
- For $k = \mathbb{R}$, separable algebras are those of the form $\mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{C} \times \cdots \times \mathbb{C}$.

Constructing a Locally Connected Geometric Morphism

- We have a full subcategory $\mathscr{D}$ of connected component objects in $\mathscr{C}$. For k-Zar we use $\mathscr{D} = k\text{-Sep}^{\mathrm{op}}$, and $\mathscr{C} = k\text{-Alg}^{\mathrm{op}}$.
- We can construct a topos out of $\mathscr{D}$ with a coverage restricted from $\mathscr{C}$.
- When does this give us a locally connected geometric morphism, whose connected components functor preserves finite products?
- We can construct a geometric morphism via Kan-extensions. A fibration of sites allows us to construct such a geometric morphism.

Fibration of Sites

- In order to be able to construct a locally connected and local geometric morphism we will use a special functor between sites.

Fibration of Sites

A fibration of sites $(\mathscr{D}, K) \to (\mathscr{C}, J)$ consists of an adjoint pair of functors $(P \dashv I)$, $P : \mathscr{D} \to \mathscr{C}$, $I : \mathscr{C} \to \mathscr{D}$ such that I is cover-preserving, I is full and faithful, and P is a categorical fibration.

- If I also reflects covers the geometric morphism is local.
- It induces a locally connected geometric morphism if P preserves covers, and it satisfies the technical condition that every K-covering sieve is J-locally connected.
- We can construct a fibration of sites from a full subcategory of our site $\mathscr{C}$. To construct a categorical fibration P we will use Categorical Galois Theory.

Categorical Galois Theory

- Categorical Galois Theory is the generalization of the Galois theory of fields to categories.
- The case we require is that of semi-left-exact reflections, which will be used to construct the necessary fibration of sites.

Semi-left-exact Reflection

Let $\mathscr{C}$ be a category with pullbacks. A reflection $(\pi \dashv i) : \mathscr{R} \to \mathscr{C}$ is semi-left-exact if for all objects C in $\mathscr{C}$, the pullback adjunction

$$(\pi_C \dashv i_C) : \mathscr{R}/(\pi C) \to \mathscr{C}/C$$

is still a reflection.

- The functor π in this definition gives us a categorical fibration.

Sifted Functors

- The functor $p_!$ is given by the left Kan extension of P. For this to preserve finite products we require P to be sifted-flat.

Sifted-Flat Functors

A small category $\mathscr{C}$ is sifted if colimits over $\mathscr{C}$ commute with finite products in Set.
A functor $P : \mathscr{C} \longrightarrow \text{Set}$ is sifted-flat if the category of elements is sifted.
A functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ is sifted-flat if for every object B in $\mathscr{B}$, the functor $\mathscr{B}(B, F-)$ is sifted-flat.

- A small category with finite coproducts is sifted.
- A functor $\pi : \mathscr{A} \longrightarrow \mathscr{B}$ which preserves finite products is sifted-flat.
- The left Kan extension of a sifted-flat functor preserves finite products.

The Result for the Zariski Topos

- Now we apply the previous results to the Zariski topos, say $k = \mathbb{R}$.
- The separable algebras are those of the form $\mathbb{R}^n \times \mathbb{C}^m$.
- The components functor π sends every algebra to the largest separable subalgebra. We have an adjunction $(i \dashv \pi)$.
- From Galois theory this is a (co-) semi-left-exact reflection.
- The components functor π preserves tensor products.
- Taking the opposite categories, we get a fibration of sites, where the left adjoint π preserves finite products, hence is sifted-flat. This fibration of sites is enough to give us a locally connected and local geometric morphism.
- This gives us the petit topos of presheaves over $(\mathbb{R} \to \mathbb{C}^{\circlearrowleft})$.
- For a general field, we get a petit topos of presheaves over the category of Galois field extensions of k.

Application: The Homotopy Category

- Having a connected components functor allows us to define a homotopy category.
- This is a locally internal category over $\mathscr{P}$.

Homotopy Category

Objects:	Objects of $\mathscr{G}$.
Homset:	$[A, B] = p_!(B^A)$.
Composition:	There is a morphism $B^A \times C^B \to C^A$ in $\mathscr{G}$ which internalizes composition. Applying $p_!$ we get:

$$p_!(B^A \times C^B) \cong p_!(B^A) \times p_!(C^B) \to p_!(C^A)$$
$$[A, B] \times [B, C] \to [A, C]$$

Conclusion

- Gros Topos: Topos of geometric objects
- Petit Topos: Topos associated to an object of a Gros Topos
- Petit Topos associated to the terminal object can be used to measure geometric points.
- If there is enough structure, this can also measure connected components.