Categorification of quantum groups

In honour of Martin Hyland and Peter Johnstone

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Available at http://www.math.columbia.edu/∼lauda/talks/

There is a Lie algebra decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

This gives a corresponding decomposition of the universal enveloping algebra

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \oplus U(\mathfrak{h}) \oplus U(\mathfrak{n}_+)$$

Taking the quantum deformation we have

$$U_q(\mathfrak{g}) = U_q^- \oplus U_q(\mathfrak{h}) \oplus U_q^+$$

U^+ has the structure of a bialgebra: try to categorify the bialgebra U_q^+

We will define a ring R so that the Grothendieck ring $K_0(R)$ of the category of finitely-generated projective R modules is isomorphic to U_q^+

$U^+ \subset U_q(\mathfrak{g})$

$\mathfrak{g} = \mathfrak{sl}_n$

$$E_i = e_{i,i+1} = \begin{array}{cc} & \begin{array}{cccc} & & i+1 & \end{array} \\ i & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & & \ddots & \\ 0 & & \dots & 0 \end{pmatrix} \end{array}$$

Lie algebra relations:

$$[E_i, E_j] = 0 \quad |i-j| > 1 \qquad\qquad [E_i, [E_i, E_j]] = 0 \quad |i-j| = 1$$

Enveloping algebra relations for $U^+(\mathfrak{sl}_n)$

$$E_iE_j = E_jE_i \qquad |i-j| > 1$$

$$2E_iE_jE_i = E_i^2E_j + E_jE_i^2 \qquad j = i \pm 1$$

Quantum enveloping algebra $U_q^+(\mathfrak{sl}_n)$

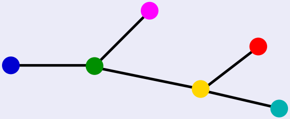
$$E_iE_j = E_jE_i \qquad |i-j| > 1$$

quantum 2 $\rightarrow (q + q^{-1})E_iE_jE_i = E_i^2E_j + E_jE_i^2 \qquad j = i \pm 1$

$U^+(\mathfrak{sl}_n)$ has a generator E_i for each vertex of the Dynkin graph

U^+ for any Γ

Let Γ be an unoriented graph with set of vertices I.

$\Gamma =$ 

U^+ is the $\mathbb{Q}(q)$-algebra with:

- generators: $E_i \quad i \in I$
- relations: $E_iE_j = E_jE_i$ if 

  $(q+q^{-1})E_iE_jE_i = E_i^2E_j + E_jE_i^2$ if 

U^+ is $\mathbb{N}[I]$ graded with $\deg(E_i) = i$.

Integral form of U^+

Define quantum integers and quantum factorials:

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}} \qquad\qquad [a]! := [a][a-1]\dots[1]$$

Example

- $[1] \;=\; 1$
- $[2] \;=\; \frac{q^2-q^{-2}}{q-q^{-1}} \;=\; q + q^{-1}$
- $[3] \;=\; \frac{q^3-q^{-3}}{q-q^{-1}} \;= q^2 + 1 + q^{-2}$

The algebra $U^+_{\mathbb{Z}}$ is the $\mathbb{Z}[q,q^{-1}]$-subalgebra of U^+ generated by all products of quantum divided powers:

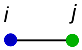
$$E_i^{(a)} := \frac{E_i^a}{[a]!}$$

Since

$$E_i^{(2)} = \frac{E_i^2}{q+q^{-1}}$$

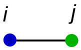
we can write the U^+ relation

$$(q+q^{-1})E_iE_jE_i = E_i^2E_j + E_jE_i^2 \quad \text{if}$$



as

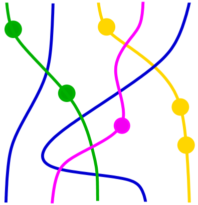
$$E_iE_jE_i = E_i^{(2)}E_j + E_jE_i^{(2)} \quad \text{if}$$



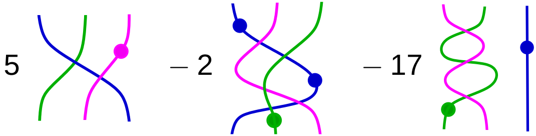
Categorification of U^+

Associated to graph Γ consider braid-like diagrams with dots whose strands are labelled by the vertices $i \in I$ of the graph Γ.

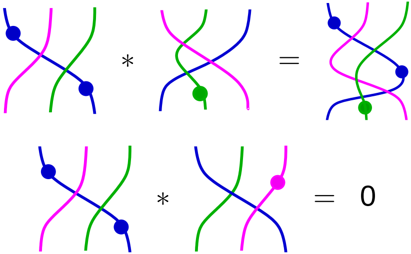
Let $\nu = \sum_{i \in I} \nu_i \cdot i$, for $\nu_i = 0, 1, 2, \ldots$
ν keeps track of how many strands of each color occur in a diagram



Form an abelian group by taking $\mathbb{Z}$-linear (or $\Bbbk$-linear) combinations of diagrams:



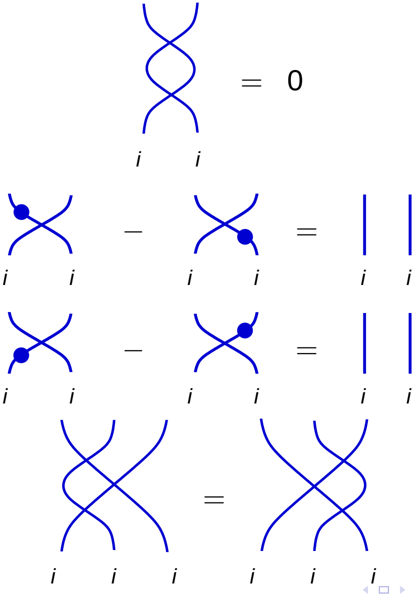
Multiplication is given by stacking diagrams on top of each other when the colors match:



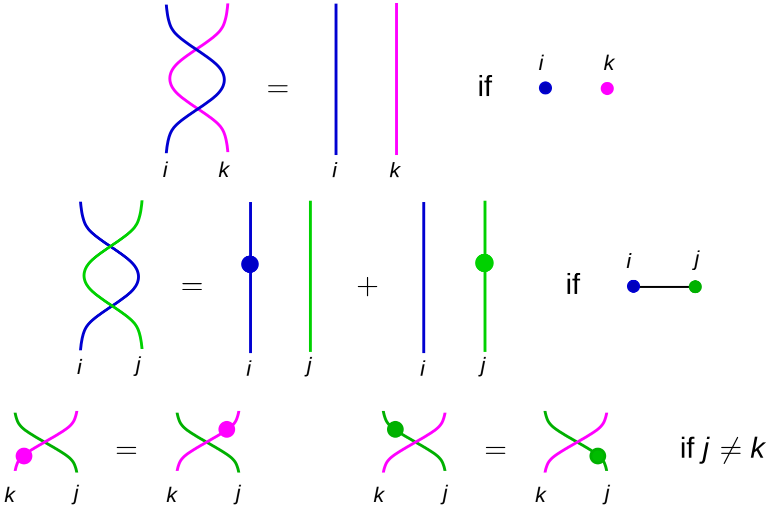
Definition

Given $\nu \in \mathbb{N}[I]$ define the ring $R(\nu)$ as the set of planar diagrams colored by ν, modulo planar braid-like isotopies and the following local relations:

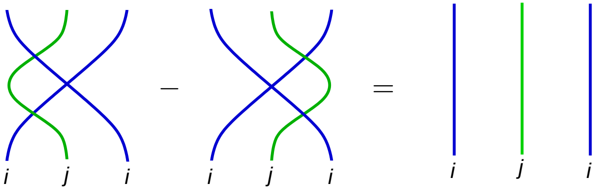
Local relations I



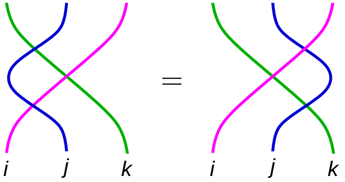
Local relations II



Local relations III



if 



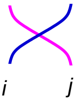
otherwise,

some of i, j, k may be equal

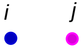
Grading

$q \longrightarrow$ grading shift

deg () $= \quad 2$

deg ()

$=$

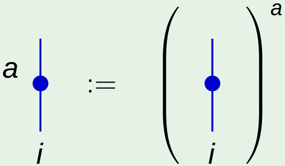
- -2 if $i = j$
- 0 if 

- 1 if 

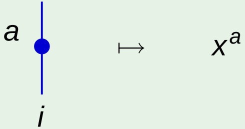
The $R(\nu)$ relations are homogeneous with respect to this grading.

Example

- If $\nu = 0$ then $R(0) = \mathbb{Z}$ with unit element given by the empty diagram.
- If $\nu = i$ for some vertex i, then a diagram is a line with some number $a \geq 0$ of dots on it.

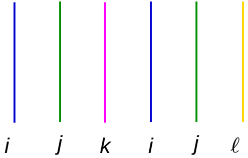


Hence, $R(i) \cong \mathbb{Z}[x]$ where the isomorphism maps



Let $R = \bigoplus_{\nu} R(\nu)$. For each product of E_i's in U^+ we have an idempotent in R:

$$E_iE_jE_kE_iE_jE_\ell \quad \mapsto \quad 1_{ijkij\ell} \; :=$$

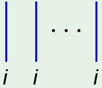


This gives rise to a projective module

$$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k\mathcal{E}_i\mathcal{E}_j\mathcal{E}_\ell \quad := \quad R1_{ijkij\ell} \quad = \quad R(2i+2j+k+\ell)1_{ijkij\ell}$$

corresponding to the idempotent $1_{ijkij\ell}$ above.

Example

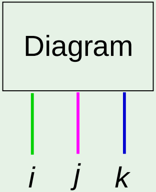
For a given $i \in I$ we write $\mathcal{E}_i^m$ for the projective module $R(mi) \cong \mathrm{NH}_m$ corresponding to the idempotent $1_{i^m} =$  , where $i^m := i \dots i$.

Example

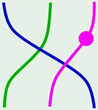
Consider

$$R1_{ijk} \quad = \quad R(i+j+k)1_{ijk}$$

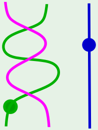
The projective module $\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k := R(i+j+k)1_{ijk}$ consists of linear combinations of diagrams that have the sequence ijk at the bottom



i.e.

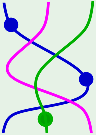


and



$\in R(i+j+k)1_{ijk}$

But

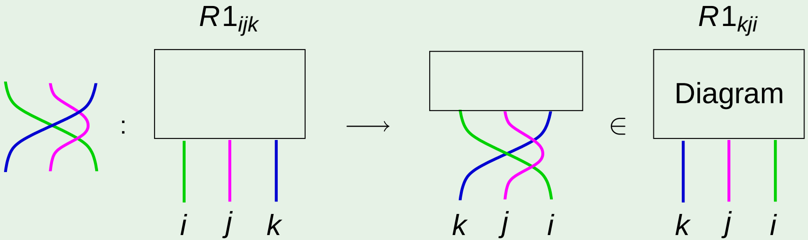


$\notin R(i+j+k)1_{ijk}$

We can construct maps between projective modules by adding diagrams at the *bottom*

Example

We get a module map from $\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k := R(i+j+k)1_{ijk}$ to $\mathcal{E}_k\mathcal{E}_j\mathcal{E}_i := R(i+j+k)1_{kjl}$ as follows:



Given a graded module M and a Laurent polynomial $f = \sum f_a q^a \in \mathbb{Z}[q, q^{-1}]$ write

$$M^{\oplus f} \qquad \text{or} \qquad \bigoplus_f M$$

to denote the direct sum over $a \in \mathbb{Z}$ of f_a copies of $M\{a\}$

Example

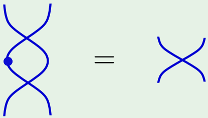
Since $[3] = q^2 + 1 + q^{-2} \in \mathbb{Z}[q, q^{-1}]$, for a graded module M

$$\bigoplus_{[3]} M = M\{2\} \oplus M\{0\} \oplus M\{-2\}$$

Example ($n = 2$)

$$E_i^{(2)} = \frac{E_i^2}{q+q^{-1}} \quad \text{or} \quad E_i^2 = (q+q^{-1})E_i^{(2)}$$

Recall that



so that $e_2 =$  is an idempotent.

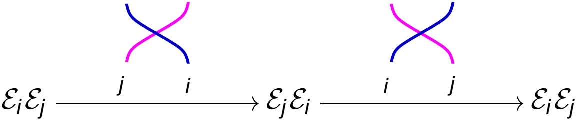
$\mathcal{E}_i^{(2)}$ is the projective module for this idempotent

$$\mathcal{E}_i^{(2)} \;:=\; R(2i)e_2\{1\}$$

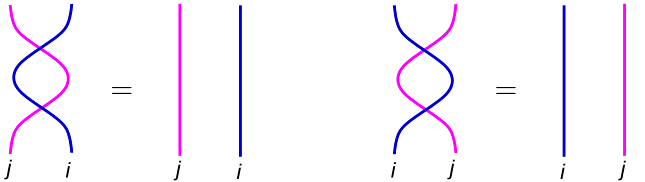
$$\mathcal{E}_i^2 \cong \mathcal{E}_i^{(2)}\{1\} \oplus \mathcal{E}_i^{(2)}\{-1\}$$

Categorification of $E_iE_j = E_jE_i$

$E_iE_j = E_jE_i$ if $\overset{i}{\bullet} \quad \overset{j}{\bullet}$ $\rightsquigarrow$ $\mathcal{E}_i\mathcal{E}_j \cong \mathcal{E}_j\mathcal{E}_i$ if $\overset{i}{\bullet} \quad \overset{j}{\bullet}$

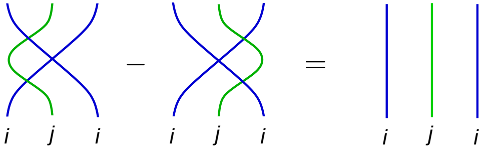


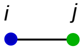
These maps are isomorphisms since

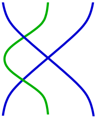


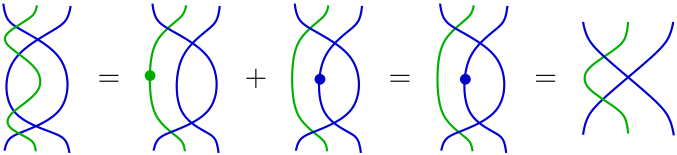
if $\overset{i}{\bullet} \quad \overset{j}{\bullet}$

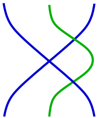
Categorification of $E_iE_jE_i = E_i^{(2)}E_j + E_jE_i^{(2)}$



if 

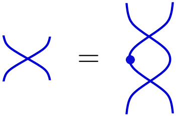
Let $e' =$ 

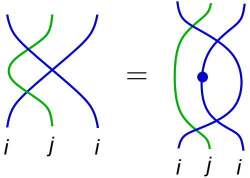
$(e')^2 =$  $= e'$

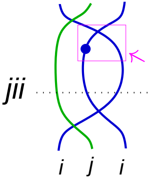
$e'' = 1_{iji} - e' = -$  is idempotent too $\quad (e'')^2 = e''$

Orthogonality $e'e'' = e''e' = 0$ and $1_{iji} = e' + e''$ imply

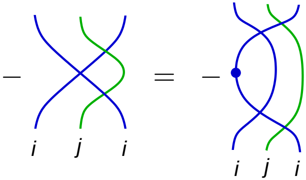
$$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \cong \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i e' \oplus \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i e''$$

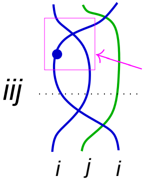
But  $\Rightarrow$

$e' =$ 



idempotent e_2 of projection onto $\mathcal{E}_i^{(2)}$

$e'' = -$ 

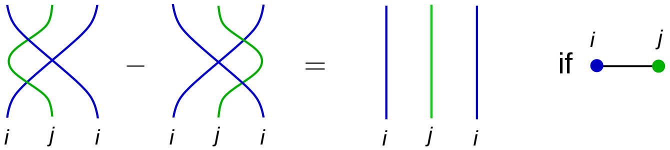


idempotent e_2 of projection onto $\mathcal{E}_i^{(2)}$

Therefore,

$$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i e' \cong \mathcal{E}_j\mathcal{E}_i^{(2)} \qquad \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i e'' \cong \mathcal{E}_i^{(2)}\mathcal{E}_j$$

so that the relation



together with the other relations imply

$$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \cong \mathcal{E}_j\mathcal{E}_i^{(2)} \oplus \mathcal{E}_i^{(2)}\mathcal{E}_j$$

Grothendieck groups

$$R = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu) \qquad K_0(R) := \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu))$$

where $K_0(R(\nu))$ is the Grothendieck group of the category $R(\nu)-\mathrm{pmod}$ of graded projective finitely-generated $R(\nu)$-modules.

$K_0(R(\nu))$ has generators $[M]$ over all objects of $R(\nu)-\mathrm{pmod}$ and defining relations

$$\begin{aligned} [M] &= [M_1] + [M_2] && \text{if } M \cong M_1 \oplus M_2 \\ [M\{s\}] &= q^s [M] && s \in \mathbb{Z} \end{aligned}$$

$K_0(R(\nu))$ is a $\mathbb{Z}[q, q^{-1}]$-module.

There are induction and restriction functors corresponding to inclusions $R(\nu) \otimes R(\nu') \subset R(\nu + \nu')$

$$\mathrm{Ind}_{\nu,\nu'}^{\nu+\nu'} : R(\nu) \otimes R(\nu')-\mathrm{pmod} \to R(\nu + \nu')-\mathrm{pmod}$$

$$\mathrm{Res}_{\nu,\nu'}^{\nu+\nu'} : R(\nu + \nu')-\mathrm{pmod} \to R(\nu) \otimes R(\nu')-\mathrm{pmod}$$

Summing over all ν, ν' gives functors

$$\mathrm{Ind} \colon (R \otimes R)-\mathrm{pmod} \to R-\mathrm{pmod} \qquad \mathrm{Res} \colon R-\mathrm{pmod} \to (R \otimes R)-\mathrm{pmod}$$

These map projectives to projectives $\Rightarrow$

$$[\mathrm{Ind}] \colon K_0(R) \otimes K_0(R) \to K_0(R) \qquad [\mathrm{Res}] \colon K_0(R) \to K_0(R) \otimes K_0(R)$$

Write $[\mathrm{Ind}](x_1, x_2)$ for $x_1, x_2 \in K_0(R)$ as $x_1 x_2$

Work over a field $\mathbb{k}$.

Theorem (A. L., M.Khovanov arXiv:0803.4121)

There is an isomorphism of twisted bialgebras:

$$\begin{aligned} \gamma\colon U_{\mathbb{Z}}^{+} &\longrightarrow K_0(R) \\ E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} \dots E_{i_k}^{(a_k)} &\mapsto \left[\mathcal{E}_{i_1}^{(a_1)} \mathcal{E}_{i_2}^{(a_2)} \dots \mathcal{E}_{i_k}^{(a_k)} \right] \end{aligned}$$

multiplication $\mapsto$ multiplication given by $[\mathrm{Ind}]$

comultiplication $\mapsto$ comultiplication given by $[\mathrm{Res}]$

The semilinear form on $U_{\mathbb{Z}}^{+}$ maps to the HOM form on $K_0(R)$

$$(x, y) = (\gamma(x), \gamma(y))$$

Injectivity of γ

Injectivity of the map $\gamma \colon U^+_{\mathbb{Z}} \to K_0(R)$ uses that U^+ is the quotient of a free associative algebra by the radical of the semilinear form. This follows from the quantum version of the Gabber-Kac theorem (proof, due to Lusztig for an arbitrary graph, uses perverse sheaves).

Surjectivity of γ

Surjectivity follows by mirroring the work of Kleshchev, Grojnowski and Vazirani.

A. L., M.Khovanov

This this theorem has an extension to the non-simply laced case. The basis of indecomposable gives a new basis for $U^+_{\mathbb{Z}}$ where structure constants are necessarily positive.

Conjecture

When the graph Γ is a tree and the field $\Bbbk = \mathbb{C}$, then under the map $U^+_{\mathbb{Z}} \to K_0(R)$ Lusztig-Kashiwara canonical basis elements go to symbols of indecomposable projectives in the Grothendieck group. If Γ contains an odd cycle then one should consider the 'sign modified' version of the rings $R(\nu)$.

arXiv:0901.4450

Brundan and Kleshchev gave an algebraic proof when Γ is a chain or a cycle.

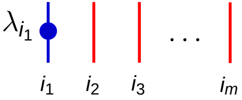
arXiv:0901.3992

The general case (over $\mathbb{C}$) was recently proven by Varagnolo and Vasserot who showed that rings $R(\nu)$ in the simply-laced case were isomorphic to certain Ext-algebras of Perverse sheaves.

Cyclotomic quotients

For a given weight $\lambda = \sum_{i \in I} \lambda_i \cdot i$ define the cyclotomic quotient R_ν^λ of $R(\nu)$ by imposing the additional relations: for any sequence $i_1 i_2 \cdots i_m$ of vertices of Γ

λ_{i_1} dots on the first strand of any sequence is zero $\longrightarrow$



$= \quad 0$

This is analogous to taking the Ariki-Koike cyclotomic quotient of the affine Hecke algebra:

$$H_d^\lambda := H_d \Big/ \left\langle \prod_{i \in I} (X_1 - q^i)^{\lambda_i} \right\rangle$$

Conjecture

For sufficiently nice Γ and $\Bbbk$, the category of finitely-generated graded modules over the ring

$$R^{\lambda} = \bigoplus_{\nu \in \mathbb{N}[I]} R^{\lambda}_{\nu}$$

categorifies the integrable version of the representation V_λ of $U_q(\mathfrak{g})$ of highest weight λ.

arXiv:0808.2032

Brundan and Kleshchev proved this conjecture in type A. Their work gives rise to an interesting new grading on blocks of the of the symmetric group. Leads to graded Specht module theory, see Brundan-Kleshchev-Wang, arXiv:0901.0218