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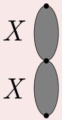

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$\Phi \cong M \otimes -$,where $M : \mathscr{A}^{op} \times \mathscr{A} \longrightarrow$ Set is a flat module.

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$$\Phi(X) = \coprod_n P_n \times X^n$$

Tom's description of the terminal coalgebra can be restated as: $T : \mathscr{A} \longrightarrow$ Set is the colimit of the diagram

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Lemma: Assume that for every finite non-empty diagram in $\mathrm{Complex}(M)$ we have limits "levelwise", for all $n \geq 0$. Then $\mathrm{Complex}(M)$ is cofiltered.

Corollary: (i) Every finitary endofunctor of an l.f.p category has a terminal coalgebra.
(ii) Freyd's endofunctor has a terminal coalgebra.

Sketch of the Proof:

Let $(a_\bullet, m_\bullet)$, $(a'_\bullet, m'_\bullet)$ in Complex(M) be a discrete diagram, i.e

$$\ldots a_n \xrightarrow{m_n} a_{n-1} \longrightarrow \cdots \longrightarrow a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0$$

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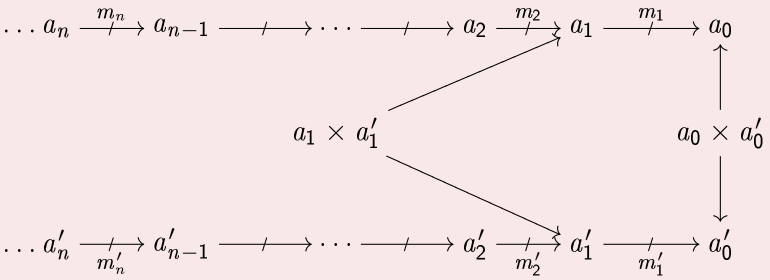

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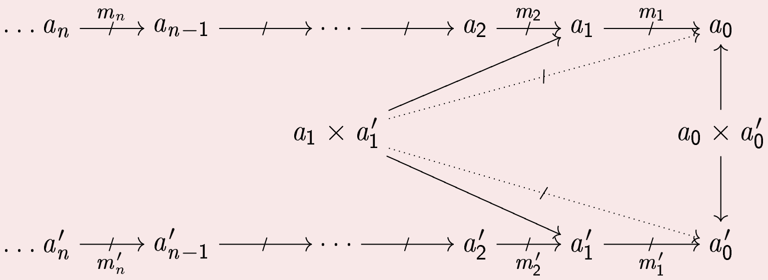

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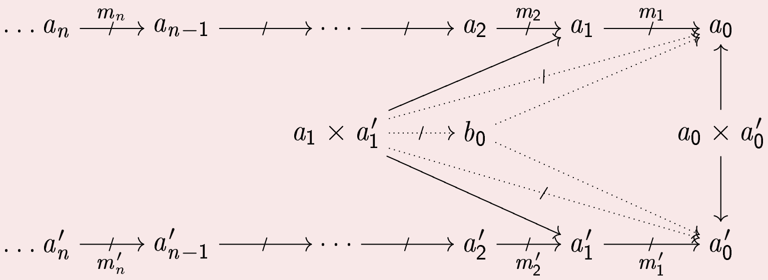

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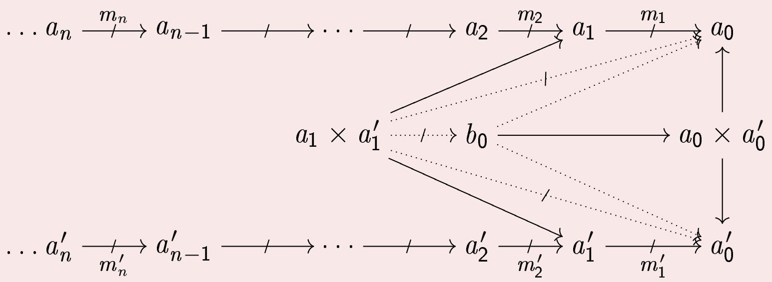

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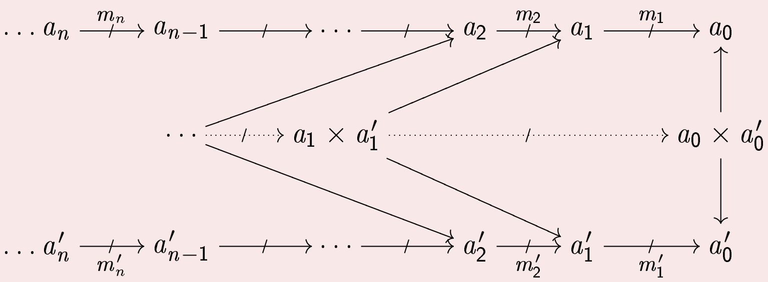

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E.g $3 \underset{d}{\overset{u}{\rightrightarrows}} 2$ with $u(\mathrm{middle}) = 1$, $d(\mathrm{middle}) = 0$ can not be coequalized.

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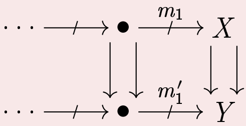

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(i) Cones exist "at the head" of finite diagrams of complexes

(ii) A technical finiteness condition holds, that allows us to infer the existence of cones at the level of complexes (not just at the head), using a topological version of König's Lemma due to A. Stone
This way

• Tom's modules for topological self-similarity become examples

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