

A Brief Complicial Tour

Dominic Verity

Centre of Australian Category Theory
Macquarie University
Sydney, Australia

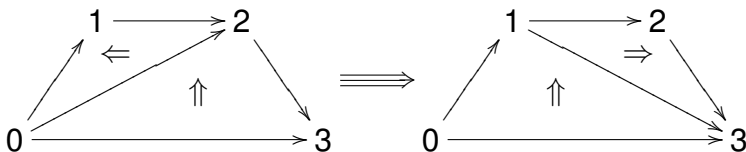
5th April 2009

Simplicial sets are lovely objects about which algebraic topologists know a lot. If something is described as a simplicial set, it is ready to be absorbed into topology. Or, in other words, no matter which definition of weak ω -category eventually becomes dominant, it will be valuable to know its simplicial nerve.

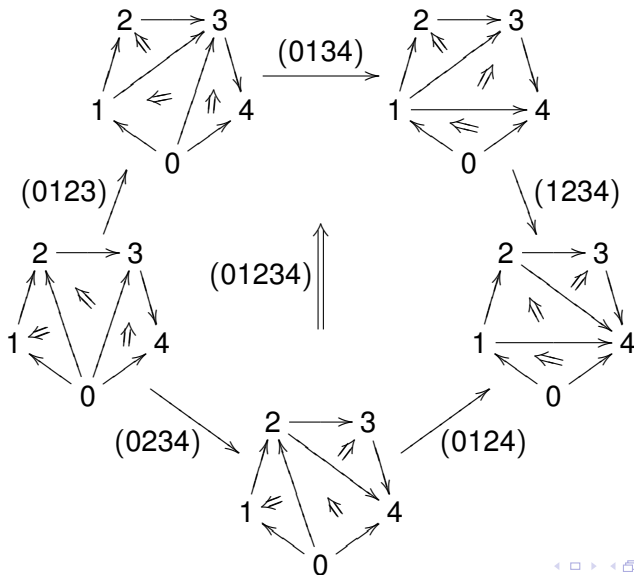
Ross Street, 2003

Nerves of ω -Categories (1)

A 3-simplex in a (strict) ω -Category



Nerves of ω -Categories (2)



A 4-simplex

Why are complicial sets of interest?

Theorem

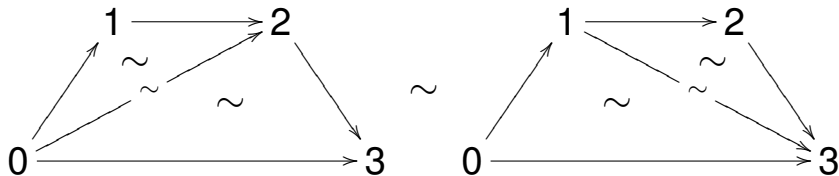
Ross Street's ω -categorical nerve construction provides an equivalence between the categories $\omega\text{-Cat}$ of (strong) ω -categories and ω -functors and Comp_S of (strong) complicial sets and stratified simplicial maps.

(Memoirs of the AMS, vol. 193, no. 905)

QUESTION: Are (strong) complicial stratifications really extra structure, or might we infer them from the underlying simplicial set itself?

ANSWER: We can construct a simplicial set supporting many distinct Roberts stratifications.

The “2-of-6” stratified simplicial set T :



Theorem

If A is a weak complicial set then there exists a category \mathbb{C} which is

- enriched in stratified simplicial sets with respect to the cartesian product (Gray tensor), and*
- has homsets which are themselves weak complicial sets*

and whose homotopy coherent nerve $N_{hc}(\mathbb{C})$ is homotopy equivalent to A .

Corollary

If A is a weak complicial set then so is its equivalence saturation $\text{Sat}(A)$

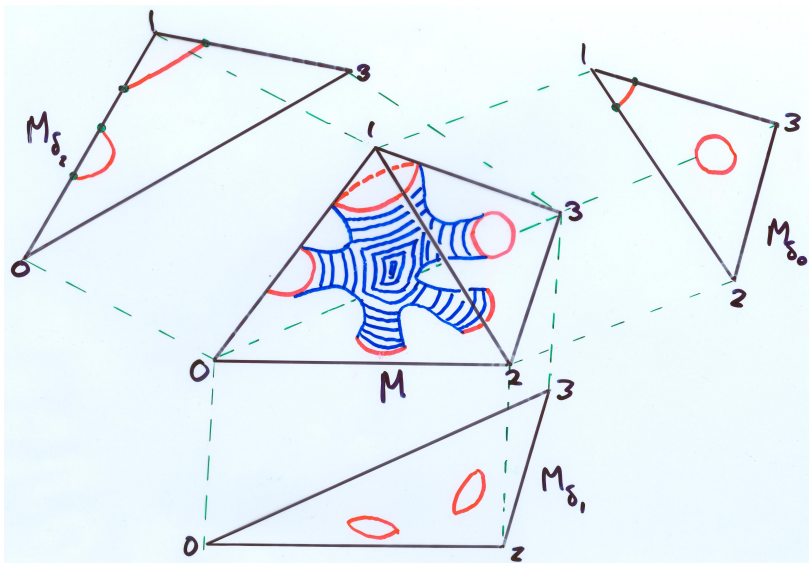
Some Observations

- If A is a weak complicial set which is
 - equivalence saturated, and
 - n -trivial for some n

then its stratification tA is maximal in the poset of all complicial stratifications on the underlying simplicial set of A .

- A simplicial set can possess no more than one stratification which makes it into an equivalence saturated and 2-trivial weak complicial set.

A Simplicial Cobordism



A nested tower of stratifications on Cob

- Cob_k the Kan-stratified simplicial set of simplicial cobordisms. This is equivalence saturated.
- Cob_t the *trivial cobordism* stratified simplicial set of simplicial cobordisms. This is **not** equivalence stratified - there exist 4 dimensional invertible cobordisms which are not trivial.
- $Cob_e \stackrel{\text{def}}{=} \text{Sat}(Cob_t)$ whose thin simplices might be called *quasi-invertible cobordism*. This is equivalence stratified.

How do we know that the stratification of Cob_e is distinct from the Kan-stratification of Cob_k ?

Theorem

The category Strat supports a model category structure whose fibrant objects are equivalence saturated weak complicial sets.

The category Strat–Cat of stratified set enriched categories (with respect to the cartesian product) also has a model category structure, whose fibrant objects are those Strat-categories whose homsets are equivalence saturated weak complicial sets.

This is closely related to the Bergner-Dwyer-Kan structure on Simp–Cat.

The homotopy coherent nerve functor provides us with a Quillen equivalence between these two Quillen model categories.