Terminal coalgebras

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June 2008

Plan

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1. Introduction to terminal coalgebras

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2. Some theory of terminal coalgebras

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3. Trimble-like n-categories

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4. Trimble-like ω-categories via terminal coalgebras

1. Introduction to terminal coalgebras

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satisfying no axioms.

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$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & B \\ \downarrow & & \downarrow \\ FA & \xrightarrow[\;Fh\;]{} & FB \end{array}$$

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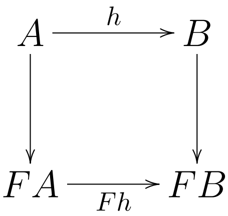

so we can look for terminal coalgebras.

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$$(m_1, m_2, m_3, \ldots)$$

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is a canonical isomorphism.

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we need to produce an infinite word in M.

1. Introduction to terminal coalgebras

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m_1	a_1

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$$a \mapsto (m_1, m_2, m_3, m_4, \dots)$$

2. Some theory of terminal coalgebras

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then f is an isomorphism.

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provided there is a terminal object 1, the limit exists, F preserves it

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which does indeed give infinite words in M.

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We note that Lambek's Lemma holds:

$$\omega\text{-}\mathbf{Cat} \cong (\omega\text{-}\mathbf{Cat})\text{-}\mathbf{Cat}.$$

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where each morphism here is truncation.

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Aim

—to apply this to Trimble's version of weak n-categories.

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So we need to build weak ω-categories from

"incoherent n-categories"

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What does "weak" mean?

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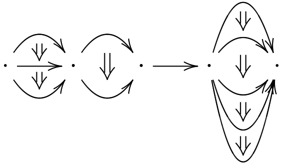

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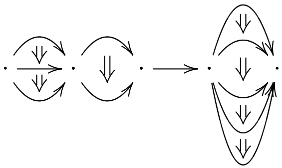

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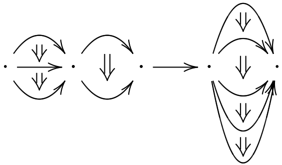

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A $(\mathcal{V}, P)$-category is defined to be a cross between

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—The underlying data is a $\mathcal{V}$-graph but composition is like a P-algebra action.

- Composition in an ordinary $\mathcal{V}$-category:

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But what operads P_n are we going to use?

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and this turns out to be easy by induction.

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"do Π locally on the hom objects"

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  $$\Pi_0 \;:\; \mathbf{Top} \longrightarrow \mathbf{Set}$$

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  points (replacing ~~the set of connected components~~)

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Objects are pairs $(\mathcal{V}, \Pi)$ where

- $\mathcal{V}$ is a category with finite products
- Π is a functor $\mathbf{Top} \longrightarrow \mathcal{V}$ preserving finite products.

Morphisms are the obvious commuting triangles.

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The terminal coalgebra is indeed the limit we were looking for.